IEOR 4106: Introduction to Operations Research: Stochastic Models

Spring 2005, Professor Whitt, First Midterm Exam

Chapters 1-4 in Ross, Thursday, February 17, 11:00am-1:00pm

Open Book: but only the Ross textbook plus one 8×11 page of notes

Justify your answers; show your work.

1. Coins (10 points)

The instructor picks out three students who have coins: Chavalit, Kuan-Ling and Xiao-Lu. Chavalit has a nickle and two dimes; Kuan-Ling has two dimes and three quarters; and Xiao-Lu has three pennies, three dimes and three quarters. Suppose that the instructor picks one of the three students at random, with each student having equal probability of being chosen. Then suppose the instructor picks one of the selected person's coins at random, with each coin having equal probability of being selected.

(a) What is the probability that the selected coin is a dime?

(b) Suppose that the selected coin is a dime. Given that the selected coin is a dime, what is the probability that the dime belonged to Kuan-Ling?

2. Bivariate Distributions (25 points)

Let X and Y be two random variables having joint probability density function (pdf)

$$f_{X,Y}(x,y) = x + y, \quad 0 \le x \le 1, \quad 0 \le y \le 1 \ .$$

(a) Determine the pdf of the random variable X.

(b) Determine the cumulative distribution function (cdf) of the random variable X.

(c) Are the random variables X and Y independent? Why or why not?

(d) Give an expression for $E[(X+Y)^2]$. (It is not necessary to calculate the number itself.)

(e) Give an expression for $P(X+Y \le 1/2)$. (It is not necessary to calculate the number itself.)

3. Mind Over Matter (15 points)

Christina claims that she has supernatural powers. She claims that she has the ability, by the power of her mental concentration, coupled with divine guidance, to increase the chance that two coins tossed together come out the same, either both heads or both tails. We decide to test Christina's talent. Suppose that we conduct a series of experiments. In each experiment we toss two coins, and observe whether or not the outcomes are the same. Suppose that we repeat the experiment $10,000$ times. Suppose that Christina is successful 5271 times out of $10,000$. Does that result present strong evidence of her abilities? Why or why not?

4. A Finite Markov Chain (30 points)

Consider a Markov chain on the ten states $\{1, 2, \cdots, 10\}$ with transition matrix P given by

$$P = \begin{pmatrix} 0.2 & 0.0 & 0.1 & 0.2 & 0.0 & 0.2 & 0.1 & 0.0 & 0.2 & 0.0 \\ 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.1 & 0.4 & 0.1 & 0.1 & 0.0 & 0.0 & 0.2 & 0.1 \\ 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \end{pmatrix}$$

**Note that we are numbering the states 1, 2, ... , 10.

(a) Which states are accessible from state 6?

(b) From which states is state 6 accessible?

(c) Do states 1 and 6 communicate?

(d) Identify the communication classes for this Markov chain.

(e) Which communication classes are closed? Which are open?

(f) Which states are transient? Which states are recurrent?

(g) Compute the six-step transition probability $P^{(6)}_{2,7}$

(h) Compute the two-step transition probability $P^{(2)}_{2,8}$

(i) Compute the two-step transition probability $P^{(2)}_{1,9}$

(j) Starting in state 2, what is the expected total number of visits to state 2?

(k) Starting in state 1, what is the expected total number of visits to state 10?

(l) Starting in state 2, what is the expected number of steps before you return again to state 2?

(m) Put the transition matrix in canonical form.

(n) Starting in state 1, what is the expected total number of visits to state 6? (Give a number if you can, but at least give a good formula.)

(o) What is the approximate value of $P^{(25)}_{1,7}$? (Give a number if you can, but at least give a good formula.)

5. A Random Walk on a Graph (20 points)

Consider a random walk on the graph consisting of 8 nodes and 10 arcs (or edges), shown in Figure 1 on the next page. There is a weight on each arc, as shown in Figure 1. Suppose that we move from a node to a neighboring node (along a connecting arc) in each step, with the neighboring node selected at random with probability proportional to the weights on the arcs

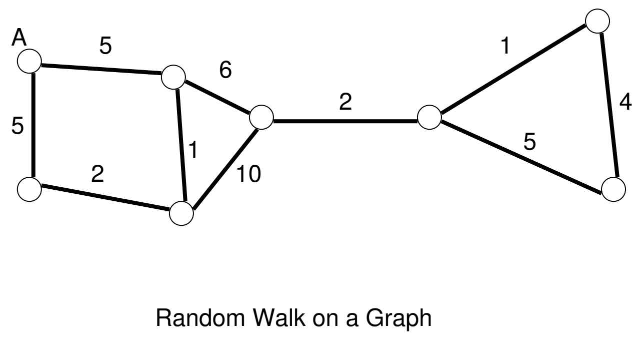

Figure 1: A random walk on a graph

out of the node. Let the random walk start in node A. Thus, the probability of moving next to each of the two neighboring nodes to node A is $5/10 = 1/2$. Let each move be a random experiment depending on the current node, but independent of the prior history.

(a) Starting in node A, what is the probability of being in node A again after *two* steps? (You need not do the arithmetic to simplify the numerical answer.)

(b) Starting in node A, what is the probability of being in node A again after *three* steps? (You need not do the arithmetic to simplify the numerical answer.)

(c) Is the Markov chain *periodic*? If so, what is the period?

(d) Starting in node A, what is the probability of being in node A again after *five* steps? (You need not do the arithmetic to simplify the numerical answer.)

(e) Starting in node A, what is the expected total number of steps before node A is visited again?

(f) What is the long-run proportion of steps ending in node A?

(g) Consider any three nodes n_1, n_2 and n_3 in the graph we have been considering. Is the probability of going first from node n_1 to node n_2 according to the Markov-chain transition matrix and then going from node n_2 to node n_3 according to the the Markov-chain transition matrix (independently of the prior history), the same as the probability of going in the opposite direction, i.e., going first from node n_3 to node n_2 and then going from node n_2 to node n_1? Why or why not?