IEOR 4106: Introduction to Operations Research: Stochastic Models

SOLUTIONS to Homework Assignment 2

More Probability Review: In the Ross textbook, *Introduction to Probability Models*, read Sections 3.1-3.5 up to (not including) Example 3.24 on p. 125 (up to Example 3.22 on p. 123 of the 9th edition). To keep the reading under control, Examples 3.14-3.16 on pages 111-117 (Examples 3.13-3.15 on pages 110-116 of the 9th edition) can be omitted. (The rest of Chapter 3 has interesting material, but we will not cover it.) Problems from Chapter 3 of Ross, plus two extra ones. In all homework, show your work.

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Problem 3.3

$$p(X = i|Y = j) = \frac{P(X = i, Y = j)}{P(Y = j)} ,$$

where

$$P(Y = j) = \sum_i p(X = i, Y = j) .$$

Then

$$E[X|Y = j] = \sum_i ip(X = i|Y = j) .$$

So

$$\begin{aligned} E[X|Y = 1] &= 2 \\ E[X|Y = 2] &= \frac{5}{3} \\ E[X|Y = 3] &= \frac{12}{5} . \end{aligned}$$

Problem 3.4

For X and Y to be independent, we need

$$p(X = i, Y = j) = p(X = i)p(Y = j) \quad \text{for all} \quad i \quad \text{and} \quad j .$$

This property does not hold here. For example,
$p(X = 1) = 2/9$ and $P(Y = 1) = 5/9$, but $p(X = 1, Y = 1) = 1/9$. Note that $1/9 \neq (2/9) \times (1/9)$.

Problem 3.7

Note that

$$\begin{aligned} p(X=1,Z=1|Y=2) &= 1/5 \\ p(X=1,Z=2|Y=2) &= 0 \\ p(X=2,Z=1|Y=2) &= 0 \\ p(X=2,Z=2|Y=2) &= 4/5 \end{aligned}$$

so that

$$\begin{aligned} p(X=1|Y=2) &= 1/5 \\ p(X=2|Y=2) &= 4/5 \end{aligned}$$

and

$$E[X|Y=2] = (1 \times \frac{1}{5}) + 2 \times \frac{4}{5} = \frac{9}{5}$$

Now turn to the second question:
First

$$P(X=1|Y=2,Z=1) = 1 \ ,$$

So $E[X|Y=2,Z=1] = 1$.

Problem 3.11

First get the pdf of Y:

$$\begin{aligned} f_Y(y) &= \int_{-y}^{y} f(x,y)\,dx \\ &= \int_{-y}^{y} \frac{(y^2-x^2)e^{-y}}{8}\,dx \\ &= \frac{y^3 e^{-y}}{6} \end{aligned}$$

Recognize that the pdf of Y is gamma with parameters $\lambda = 1$ and $\alpha = 4$; see Section 2.3.3
Then get the conditional pdf

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{3}{4}\frac{(y^2-x^2)}{y^3}, \quad -y \le x \le y, \end{aligned}$$

for $y > 0$.
To check, notice that the area under the conditional pdf $f_{X|Y}(x|y)$ is indeed 1.
Now

$$
\begin{aligned}
E[X|Y=y] &= \int_{-y}^{y} x f_{X|Y}(x|y)\,dx \\
&= \int_{-y}^{y} x \frac{3}{4}\frac{(y^2-x^2)}{y^3}\,dx \\
&= \frac{3}{4}\int_{-y}^{y} [\frac{x}{y} - \frac{x^3}{y^3}]\,dx \\
&= 0
\end{aligned}
$$

You can see the result in advance by noticing that the conditional pdf $f_{X|Y}(x|y)$ is symmetric about 0.

Problem 3.12

First find the conditional density of X given that $Y = y$. See Section 3.3. The general formula is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \ .$$

To get the density of Y, we need to integrate:

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)\,dx = e^{-y} \ ,$$

which we recognize as the pdf of an exponential random variable with mean 1. Hence

$$f_{X|Y}(x|y) = \frac{e^{-x/y}e^{-y}}{ye^{-y}} = \frac{e^{-x/y}}{y} \ ,$$

which we recognize as the pdf of an exponential random variable with mean y.

Hence $E[X|Y=y] = y$.

Problem 3.14

You can see that the conditional distribution is uniform over $[0,1/2]$, so that the conditional mean must be $1/4$. You can also proceed more deliberately: Construct the conditional density:

$$f_{X|X<1/2}(x) = \frac{f_X(x)}{P(X<1/2)} = \frac{1}{1/2} = 2, \quad 0 \le x \le 1/2 \ .$$

Now compute the expected value with respect to this new density:

$$E[X|X<1/2] = \int_0^{1/2} x f_X(x)\,dx = \int_0^{1/2} 2x\,dx = \frac{1}{4}$$

Problem 3.15

Since the joint density is independent of x, you can see that the conditional density must be uniform in x over the allowed range, which is the interval $[0, y]$. Again we can proceed more deliberately: Start by computing the conditional density of X given $Y = y$:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{(1/y)e^{-y}}{f_Y(y)}, 0 < x < y,$$

where

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,dx = \int_0^y (1/y)e^{-y}\,dx = e^{-y}, \quad 0 \leq y < \infty \ .$$

Hence,

$$f_{X|Y=y}(x) = \frac{(1/y)e^{-y}}{e^{-y}} = \frac{1}{y}, 0 < x < y,$$

Then

$$E[X^2|Y = y] = \int_0^y \frac{1}{y} x^2\,dx = \frac{y^2}{3} \ .$$

Problem 3.37

$$E[X] = (1/3)2.6 + (1/3)3.0 + (1/3)3.4 = 9.0/3 = 3.0$$

Essentially the same logic applies with the second moment. Recall that the variance of a Poisson random variable equals the mean. Hence the second moment of a Poisson random variable is the mean plus the square of the mean. We use the fact that the second moment of a mixture is the mixture of the second moments:

$$E[X^2] = (1/3)[(2.6) + (2.6)^2] + (1/3)[(3.0) + (3.0)^2] + (1/3)[(3.4) + (3.4)^2] = 12.107$$

Then

$$Var(X) = E[X^2] - (EX)^2 = 12.107 - 9.000 = 3.107 \ .$$

Problem 3.40 (a)

(a) Let X denote the number of the door chosen on the first try. Let N be the total number of days spent in jail. The idea is to condition on X, using the fact that the problem repeats when the prisoner returns to his cell.

$$\begin{aligned} E[N|X = 1] &= 2 + E[N] \\ E[N|X = 1] &= 3 + E[N] \\ E[N|X = 1] &= 0 \\ E[N] &= (.5)(2 + E[N]) + (0.3)(3 + E[N]) + (0.2)(0) \ , \end{aligned}$$

from which we deduce that

$$E[N] = 9.5 \quad \text{days} \ .$$

Extra Problem 1. Suppose that you flip a fair coin $1,000,000$ times. What is the approximate probability of getting at least $501,500$ heads?

The exact distribution of the number of heads is binomial with parameters $n = 1,000,000$ and $p = 1/2$, but you should use a normal distribution approximation, based on the central limit theorem. The mean number of heads is $500,000$ and the standard deviation is 500. So we are asking about the probability of exceeding the mean by at least 3 standard deviations. From the table for the normal distribution on page 81, we see that the probability is $1 - 0.9987 = 0.0013$ (about $1/1000$). The law of large numbers tells us that the proportion gets closer and closer to $1/2$ as the sample size grows. The central limit theorem allows us to be much more precise, and see how likely are actual outcomes.

Extra Problem 2. Suppose that you own shares of stock in the company Lewser, Inc. Suppose that the current (initial) stock price is \$100 per share. Suppose that we regard the daily changes in stock price as independent and identically distributed random variables (a simple additive random walk model). Suppose that the expected daily change is $-\$0.10$ or -10 cents per day (as one would expect from a stock with that name). Using units of dollars, suppose that the variance of the daily change is 0.25. Suppose that the distribution of the daily change is a gamma distribution. What is the approximate probability that the stock price has not dropped (is at least \$100) after 100 days?

The total change in 100 days is the sum of 100 IID random variables, each distributed as the single daily change. The exact distribution is complicated, but again we can apply the central limit theorem and get a normal approximation. We do not use the fact that the distribution is gamma. All we care about is that the daily change has a finite mean and variance. And of course we care about the values of the mean and variance. Let S_{100} be the change in the stock price over 100 days. We would write $S_{100} = X_1 + X_2 + \cdots + X_{100}$, where X_j is the change in day j. Here are the mean and variance: The mean of the sum is (always) the sum of the means, so that

$$E[S_{100}] = E[X_1 + X_2 + \cdots + X_{100}] = 100E[X_1] = -10 \quad \text{(dollars)}$$

and, because of the IID assumption, the variance of the sum is also the sum of the variances, so that it is

$$Var[S_{100}] = Var[X_1 + X_2 + \cdots + X_{100}] = Var[X_1] + \cdots + Var[X_{100}] = 100Var[X_1] = 25$$

so that the standard deviation of S_{100} is exactly 5 (the square root of the variance). Hence

$$\begin{aligned} P(100 + S_{100} \geq 100) &= P(S_{100} \geq 0) \\ &= P\left(\frac{S_{100} + 10}{5} \geq \frac{0 + 10}{5}\right) \\ &\approx P\left(N(0,1) \geq 2\right) = 1 - 0.9772 = 0.0228\ . \end{aligned}$$

See pages 79-83 of the book and the class lecture notes for more explanation.