Poisson Process: Special Case of Many Things

It is useful to be aware that a Poisson process is a special case of several important stochastic processes. That leads to different equivalent definitions of a Poisson process, as in Definitions 5.2 and 5.3 of the Ross text. It also leads to different ways to analyze a Poisson process.

1. A Point Process and a Counting Process

A **point process** on the positive half line, i.e., on the interval $[0, \infty)$, is a random distribution of points on the positive half line. We may specify the distribution in three ways: (i) by specifying the distribution of the locations of the points, (ii) by specifying the distribution of the intervals between successive points and (iii) by specifying the distribution of the associated counting process. Let S_n be the location of the n^{th} point, where $S_0 \equiv 0$ (without there being a 0^{th} point). Let $X_n \equiv S_n - S_{n-1}$ be the interval between the $(n-1)^{\text{st}}$ point and the n^{th} point. Let the associated **counting process** be defined by

$$N(t) \equiv \max\{k \geq 0 : S_k \leq t\}, \quad t \geq 0.$$

In other words, a point process may be specified in three ways, via the stochastic processes: (i) $\{S_n : n \geq 0\}$, (ii) $\{X_n : n \geq 1\}$ and (iii) $\{N(t) : t \geq 0\}$. The first representation $\{S_n : n \geq 0\}$ is the typical form for a **point process**. The last representation $\{N(t) : t \geq 0\}$ is the typical form for a **counting process**.

A picture makes this clear; see Figure 1.

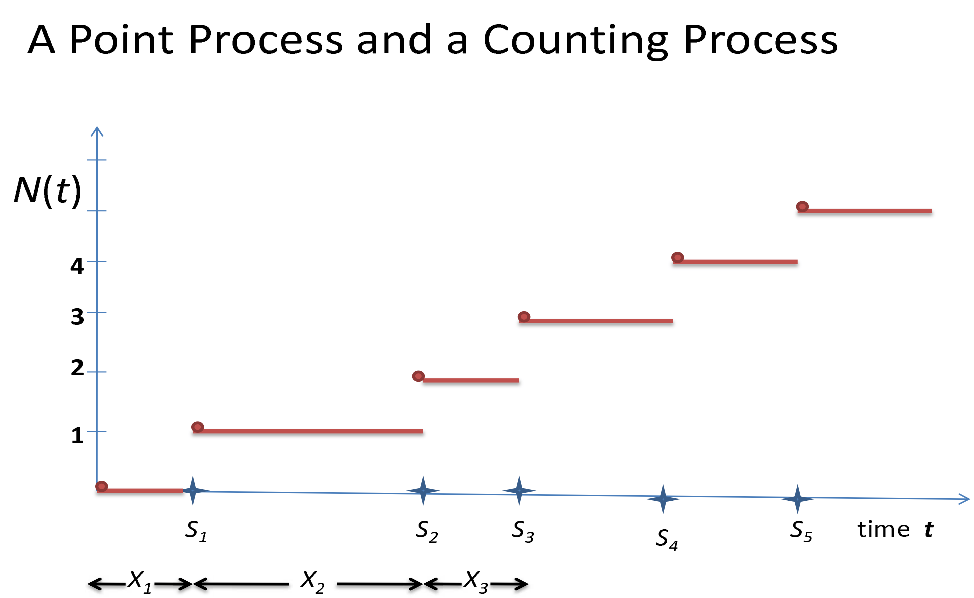

Figure 1: A Sample Path of a Counting Process.

For any point process or counting process, there is an important **inverse relation**, mentioned in Section 5.3.3 after Proposition 5.1 and discussed at greater length in (7.2) of Chapter

7. For any nonnegative n and t,

$$S_n \le t \quad \text{if and only if} \quad N(t) \ge n.$$

This is for any possible realization; i.e., it is valid with probability 1. Again, a picture makes this clear: In Figure 1, the counting process view looks at the horizontal x axis as the domain and the vertical y axis as the range (mapping time t into the number $N(t)$), while the point process view looks at the vertical y axis as the domain and the horizontal x axis as the range (mapping the nonnegative integers n into the n^{th} point S_n).

2. Alternative Definitions of a Poisson Process

(a) **Standard Definition**

The standard definition of a Poisson process is a counting process $\{N(t) : t \ge 0\}$ such that (i) $N(t)$ has a **Poisson distribution** with mean λt for each $t > 0$, where $\lambda > 0$ is some parameter, and (ii) the process has **independent increments**.

Property (i) means that

$$P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

for any nonnegative integer k (where $x^0 = 1$ and $0! = 1$). As a consequence,

$$E[N(t)] = \text{Variance}(N(t)) = \lambda t.$$

The stochastic process $N \equiv \{N(t) : t \ge 0\}$ has **independent increments** if the number of point in any number of **disjoint intervals** are independent random variables. The number of points in the interval $(a, b]$, closed on the right and open on the left, is $N(b) - N(a)$. (The probability of a point at any specific location will be 0, because the distance between successive points has a density.) An **increment** of the stochastic process $N \equiv \{N(t) : t \ge 0\}$ is $N(b) - N(a)$.

Suppose that $(t_1, t_2]$, $(t_3, t_4] \ldots (t_{2k-1}, t_{2k}]$ are k disjoint intervals, i.e., with

$$0 \le t_1 < t_2 \le t_3 < t_4 \le \cdots \le t_{2k-1} < t_{2k}.$$

Then the k random variables $N(t_2) - N(t_1)$, $N(t_4) - N(t_3), \ldots N(t_{2k}) - N(t_{2k-1})$ are mutually independent random variables. As a consequence, if $0 \le t_1 < t_2 \le t_3 < t_4$, then

$$\begin{aligned} P(N(t_2) - N(t_1) = j, N(t_4) - N(t_3) = k) &= P(N(t_2) - N(t_1) = j)P(N(t_4) - N(t_3) = k) \\ &= \frac{e^{-\lambda(t_2 - t_1)}(\lambda(t_2 - t_1))^j}{j!} \frac{e^{-\lambda(t_4 - t_3)}(\lambda(t_4 - t_3))^k}{k!} \end{aligned}$$

(b) **renewal process**

A point process (or counting process) is a renewal process if the intervals between points, i.e., the random variables X_n defined above, are independent and identically distributed (i.i.d.). In renewal theory (Chapter 7), much attention is given to the associated renewal counting process $\{N(t) : t \ge 0\}$.

A Poisson process (Chapter 5) is a special case of a renewal process (Chapter 7) in which the times between renewals have an exponential distribution. This corresponds to Proposition 5.1 in Section 5.3.

Let us compute the distribution of X_1:

$$P(X_1 > t) = P(N(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}, \quad t \ge 0.$$

Hence, the cumulative distribution function (cdf) of X_1 is

$$F_{X_1}(t) \equiv P(X_1 \le t) = 1 - e^{-\lambda t} \quad \text{(i.e., exponential with mean } 1/\lambda)$$

and the associated probability density function (pdf) is

$$f_{X_1}(t) = \lambda e^{-\lambda t}, \quad t \ge 0.$$

(c) **CTMC**

A Poisson Process is a special case of a continuous-time Markov chain (CTMC). One way to characterize a CTMC is via its infinitesimal rate matrix, usually denoted by Q. For a Poisson process, we have $Q_{i,i+1} = \lambda$, $Q_{i,i} = -\lambda$ and $Q_{i,j} = 0$ for all other j. The rate matrix Q determines the probability transition matrix $P(t) \equiv (P_{i,j}(t))$ via a matrix ordinary differential equation (ODE)

$$\dot{P}(t) = P(t)Q = QP(t) \ ;$$

see the CTMC lecture notes posted on line for March 1. That corresponds to Definition 5.3. (The stationary and independent increments property assumed there implies that Markov property.) Notice that the "little oh" notation in (iii) of definition 5.3 just means that the function has a derivative (from the right at 0).

(d) **Lévy Process**

A Poisson Process is a special case of a Lévy process. A Lévy process is a stochastic process with stationary and independent increments. Assume that it takes the value at time 0. The unique Lévy process with continuous sample paths is Brownian motion. (Surprising as it may seem, we do not need to directly assume that the increments have a normal distribution.) The unique Lévy process with sample paths having only unit jumps is the Poisson process. (Surprising as it may seem, we do not need to directly assume that the increments have a Poisson distribution.)

(e) **Poisson random measure**

A Poisson process (as well as a nonhogeneous Poisson process - Section 5.4 - can be viewed as a special case of a Poisson random measure. In the standard case, the underlying space is the positive real line $[0, \infty)$. But Poisson random measures can be defined on more general spaces, such as $\mathbb{R}^2$, corresponding to random points on the blackboard. Exercise 5.94 discusses a special case of a Poisson random measure in which the space is $\mathbb{R}^2$.

3. Basic Properties

Please pay attention to the basic properties on the Concise Summary page: (i) interarrival times (renewal process view above), (ii) thinning, (iii) superposition and (iv) conditioning.