IEOR 4106, Spring 2011, Professor Whitt

Introduction to Renewal Theory: Tuesday, April 5

1. Painful Memories: Visit to a Museum

Recall: A popular museum is open for 11 hours each day, but only admits new visitors during the first 9 hours. All visitors must leave at the end of the eleven-hour period. Suppose that visitors (which may be individuals or small groups, which we treat as individuals) arrive at the museum according to a Poisson process at rate 100 per hour. Suppose that each visitor, independently of all other visitor, spends a random time in the museum that is uniformly distributed between 0 and 2 hours. Suppose that 25% of these visitors visit the museum gift shop while they are in the museum. (Visiting the gift shop is assumed not to alter the total length of stay in the museum.) Statistics have revealed that the dollar value of the purchases by each visitor to the gift shop has, approximately, a gamma distribution with mean \$40 and standard deviation \$30.

I. Formalizing what has been assumed: random variables and stochastic processes. We can formalize the information problem by defining several stochastic processes, some sequence of random variables and some continuous-time processes. We now review the structure, based on the assumptions above:

continuous-time stochastic processes: Let $N(t)$ be the number of visitors to come to the museum in the first t hours, i.e., during the time interval $[0, t]$. (Here $0 \leq t \leq 9$, but ignore the termination time; think of it as a stochastic process with $t \geq 0$.) Let $M(t)$ be the number of visitors to come to the museum during the time interval $[0, t]$ that will go to the gift shop sometime during their visit. Let $D(t)$ be the dollar value of the purchases from the gift shop by all the visitors that initially arrived at the museum in the interval $[0, t]$.

associated random variables; Let X_i be the interarrival time between the $(i-1)^{st}$ visitor and the i^{th} visitor to the museum. Let U_j be the interarrival time between the $(j-1)^{st}$ visitor and the j^{th} visitor to the museum, counting only those that eventually go to the gift shop during their visit.. Let $Z_i = 1$ if the i^{th} visitor to the museum goes to the gift shop sometime during his visit; otherwise $Z_i = 0$. Let Y_j be the dollar value of all purchases by the $j^{\|rmth}$ visitor to arrive among those that go to the gift shop.

discrete-time stochastic processes: For the random variables above, we get the associated stochastic process (sequence of these random variables): $X_i : i \geq 1\}$, $U_j : j \geq 1\}$, $Z_i : i \geq 1\}$, $Y_j : j \geq 1\}$.

What structural properties hold for these stochastic processes? Directly, by the **assumptions above**, $N(t) : t \geq 0\}$ is directly been assumed to be a Poisson process with arrival rate $\lambda = 100$ per hour. The sequence of random variables $Z_i : i \geq 1\}$ is i.i.d. (independent and identically distributed). The sequence of random variables $Y_j : j \geq 1\}$ is also i.i.d. Those properties have been directly assumed!

From the associated **theory for those processes**, we can also conclude that the sequence $X_i : i \geq 1\}$ is i.i.d with X_i having an exponential distribution with mean $1/\lambda = 1/100$ hours. (The interarrival times of a Poisson process are exponential with a mean equal to the reciprocal of the arrival rate.) We can directly write $M(t)$ as a **compound Poisson process**:

$$M(t) = \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0.$$

However, we also recognize that the stochastic process $M(t) : t \geq 0\}$ is constructed from the stochastic process $N(t) : t \geq 0\}$ by performing independent thinning with a constant thinning probability $p = 1/4$. Hence, the stochastic process $M(t) : t \geq 0\}$ is itself a Poisson process with arrival rate $\lambda p = 100 \times (1/4) = 25$ per hour. Hence, we can also conclude that the sequence of the interarrival times of $M(t) : t \geq 0\}$, i.e., $U_j : j \geq 1\}$, is i.i.d with U_j having an exponential distribution with mean $1/\lambda p = 1/25$ hours. Next, we see that we can directly write the stochastic process $\{D(t) : t \geq 0\}$ as yet another **compound Poisson process**:

$$D(t) = \sum_{j=1}^{M(t)} Y_j, \quad t \geq 0.$$

However, the random variables Y_j do not assume only the values 1 or 0, so that the stochastic process $\{D(t) : t \geq 0\}$ is **not** itself another Poisson process. Indeed, the random variables Y_j are assumed to have a gamma distribution (although that detail played no role in the problem solution). We can understand what that means, however, by referring to Sections 2.33 and 5.2.3 in the textbook. The implications of the compound Poisson process representation are explained in Section 5.4.2 of the textbook.

II. A more general model: enter renewal processes. Suppose that the random times between arrivals to the museum, X_i no longer have an exponential distribution, but the sequence $X_i : i \geq 1\}$ is still i.i.d with X_i having some other distribution with mean $1/\lambda = 1/100$ hours. What happens now?

First, the stochastic process $N(t) : t \geq 0\}$ counting the number of arrivals to the museum over time is no longer a Poisson process, but it still has important structure. Is now is a **renewal process**, as treated in Chapter 7. The stochastic process $M(t) : t \geq 0\}$ becomes what is called a **renewal reward process**, where the random variables Z_i play the role of the rewards; see §7.4 in the book. However, since the random variables Z_i assume only the values 0 and 1, we see that the times between successive events in the stochastic process $M(t) : t \geq 0\}$, i.e., the random variables U_j are i.i.d. too, so the stochastic process $M(t) : t \geq 0\}$ itself is also a renewal process. Consequently, the stochastic process $\{D(t) : t \geq 0\}$ as yet another **renewal reward process**.

the significance: The significance of the new structure above is that we can actually answer all the questions about the stochastic process $\{D(t) : t \geq 0\}$, even though we have made the model much more general. For example, we can actually generate a new normal approximation, which necessarily is more complicated. We will focus especially on long run averages. Then the new story is relatively simple, agreeing with what happens in the special case in which $N(t) : t \geq 0\}$ is a Poisson process, but not actually using those extra assumptions. The power of this generalization is nicely illustrated by examples. Two such examples, illustrating possible career paths you might not be considering, appear below.

2. Small-Town Traffic Cop: Speeding Ticket Revenue

A small-town traffic cop spends his entire day on the lookout for speeders. The policeman cruises on average approximately 10 minutes before stopping a car for some offense. Of the cars he stops, 90% of the drivers are given speeding tickets with an \$80 fine. It takes the policeman an average of 5 minutes to write such a ticket. The other 10% of the stops are for more serious offenses, leading to an average fine of \$300. These more serious charges take an average of 30 minutes to process. In the long run what is the rate of money brought in by fines?

ANSWER

The average time between successive stops is

$$10 + (0.9 \times 5) + (0.1 \times 30) = 10 + 4.5 + 3.0 = 17.5 \quad \text{minutes} .$$

The average fine revenue per stop is

$$0.9 \times 80 + 0.1 \times 300 = 72 + 30 = 102$$

Hence, by the renewal reward theorem, the long-run average rate fine revenue is accrued is

$$\frac{E[\text{fine per stop}]}{E[\text{time per stop}]} = \frac{\$102}{17.5 \quad \text{minutes}} = \$5.28 \quad \text{per minute} .$$

Not a bad job!

3. Supporting Theory

The supporting theory is the strong law of large numbers (SLLN) for renewal reward processes, Proposition 7.3 (a) on page 439 (p. 433 of 9th ed.), which in turn draws on the SLLN for the renewal counting process, Proposition 7.1 on page 428 (p. 423 of 9th ed.), which in turn draws on the classical SLLN for sums of i.i.d. random variables, Theorem 2.1 on page 79. These SLLN's require that the random variables involved have finite means. (The word "strong" means that the limit holds "with probability 1," as opposed to some weaker notion. The weak law of large numbers (WLLN) involves convergence in probability or convergence in distribution, a weaker notion. We do not focus on the issue of the mode of convergence.) The two propositions in Section 7 are very easily proved, given the classic result stated without proof in Chapter 2. Those proofs are given in the book, and were done in class.

What that theory concludes is that **the long-run average reward over time is the expected reward per cycle, divided by the expected length of the cycle**, where a cycle is the time between successive renewals. As in Proposition 7.3, the total reward earned up to time t is defined as

$$R(t) \equiv \sum_{i=1}^{N(t)} R_i, \quad t \geq 0, \tag{1}$$

where R_i is the reward in cycle i and $\{N(t) : t \geq 0\}$ is the renewal counting process with X_i as the time between renewals $i-1$ and i. The SLLN for renewal-reward processes states that

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[R_1]}{E[X_1]}. \tag{2}$$

One step in proving the important result in (2) is establishing the SLLN for $N(t)$, which states that

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{E[X_1]}. \tag{3}$$

For the rest, we use the usual SLLN for the summands:

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} R_i}{n} = E[R_1]. \tag{4}$$

3. Long-haul Truck Driver: Driving Back and Forth

A truck driver continually drives from Atlanta (A) to Boston (B) and then immediately back from B to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is randomly chosen, uniformly in the interval $[40, 60]$. Each time he drives from B to A, he drives at a fixed speed that (in miles per hour) is randomly chosen, being either 40 or 60, each with probability 1/2. (Going from B to A the random speed has a Bernoulli distribution.) The successive random experiments at the beginning of each trip are mutually independent.

(a) In the long run, what proportion of his driving time is spent driving from A to B?

(b) In the long run, what proportion of his driving time is spent driving at a speed of 40 miles per hour?

ANSWERS

The key idea has nothing to do with this course: You should remember **dirt**. If you can't remember this, you are **" dumb as dirt"**, to quote Robin Williams in his recording of Pecos Bill (another weird reference).

Dirt means $\mathbf{D} = \mathbf{RT}$, *distance equals rate times time.* Here we are given random speeds or rates, but we ask about times. So whatever the distance D, the time T is $T = D/R$. We must thus average the reciprocal of the rate. That explains the solution. Note that $E[1/R]$ is not $1/E[R]$. That explains why the two expected times are not equal.

(a) The proportion of his driving time spent driving from A to B is

$$\frac{E[T_{A,B}]}{E[T_{A,B}] + E[T_{B,A}]} ,$$

where $E[T_{A,B}]$ is the expected time to drive from A to B, while $E[T_{B,A}]$ is the expected time to drive from B to A.

To find $E[T_{A,B}]$ and $E[T_{B,A}]$, we use the elementary formula $d = rt$ (distance = rate $\times$ time). Let S be the driver's random speed driving from A to B. Then

$$\begin{aligned} E[T_{A,B}] &= \frac{1}{20}\int_{40}^{60} E[T_{A,B}|S = s]\, ds = \frac{1}{20}\int_{40}^{60} \frac{d}{s}\, ds \\ &= \frac{d}{20}(ln(60) - ln(40)) = \frac{d}{20}(ln(3/2)\ . \end{aligned}$$

Similarly,

$$\begin{aligned} E[T_{B,A}] &= \frac{1}{2}E[T_{B,A}|S = 40] + \frac{1}{2}E[T_{B,A}|S = 60] \\ &= \frac{1}{2}(\frac{d}{40} + \frac{d}{60}) = \frac{d}{48} \end{aligned}$$

(b) Assume that a reward is earned at rate 1 per unit time whenever he is driving at a rate of 40 miles per hour, we can again apply the renewal reward approach. If p is the long-run proportion of time he is driving 40 miles per hour,

$$p = \frac{(1/2)d/40}{E[T_{A,B}] + E[T_{B,A}]} = \frac{1/80}{\frac{1}{20}ln(3/2) + 1/48}\ .$$