IEOR 4106, Spring 2011, Professor Whitt

Brownian Motion, Martingales and Stopping Times

Thursday, April 21

1 Martingales

A stochastic process $\{Y(t) : t \geq 0\}$ is a **martingale** (MG) with respect to another stochastic process $\{Z(t) : t \geq 0\}$ if

$$E[Y(t)|Z(u), 0 \leq u \leq s] = Y(s) \quad \text{for} \quad 0 < s < t \ .$$

As an extra technical regularity condition, we require that $E[|Y(t)|] < \infty$ for all t as well.

The stochastic process $\{Z(t) : t \geq 0\}$ above is giving relevant information. The segment $\{Z(s) : 0 \leq s \leq t\}$ gives the **history** up to time t. Often the information process Z is just the given stochastic process Y. Then we just say that $\{Y(t) : t \geq 0\}$ is a **martingale** (MG), without saying "with respect to."

Example 1.1 Let $B(t) : t \geq 0\}$ be **standard** ($\mu = 0$, zero-drift, and $\sigma^2 = 1$, unit variance) **Brownian motion** (BM). We will use the fact that standard BM $\{B(t) : t \geq 0\}$ is a martingale with respect to itself. Then we just say that standard BM is a martingale.

The MG property of BM, like almost everything else, is proved by applying the property of **stationary and independent increments**. We show that $E[B(t)|B(u), 0 \leq u \leq s] = B(s)$ for $0 \leq s < t$:

$$\begin{aligned}
E[B(t)|B(u), 0 \leq u \leq s] &= E[B(s) + B(t) - B(s)|B(u), 0 \leq u \leq s] \quad \text{(add and subtract)} \\
&= E[B(s)|B(u), 0 \leq u \leq s] + E[B(t) - B(s)|B(u), 0 \leq u \leq s] \\
&\qquad \text{(conditional expectation of sum is sum of conditional expectations)} \\
&= B(s) + E[B(t) - B(s)|B(u), 0 \leq u \leq s] \\
&\qquad \text{because} \quad (E[B(s)|B(u), 0 \leq u \leq s] = B(s) \\
&\qquad \text{we know } B(s)\text{, because it is in the condition, so nothing random)} \\
&= B(s) + E[B(t) - B(s)] \quad \text{(independent increments)} \\
&= B(s) + 0 = B(s) \quad \text{(BM has mean 0).}
\end{aligned}$$

But we shall be interested in martingales with respect to BM that are themselves appropriate functions of BM.

Example 1.2 An example considered below is the stochastic process $\{B(t)^2 - t : t \geq 0\}$. We let $Y(t) = B(t)^2 - t$ and $Z(t) = B(t)$ in the definition above. Thus we say that $\{B(t)^2 - t : t \geq 0\}$ is a MG with respect to BM $\{B(t) : t \geq 0\}$. However, it can be shown that $\{B(t)^2 - t : t \geq 0\}$ is also a MG with respect to $\{B(t)^2 - t : t \geq 0\}$. And similarly for other functions of BM that we will consider. They too are simply martingales (with respect to their own "internal" history), but we will simply show that they are martingales with respect to BM.

2 Stopping Times

A nonnegative random variable T is a **stopping time** relative to a continuous-time stochastic process $\{Z(t) : t \geq 0\}$ if, for any time t, the event $\{T \leq t\}$ depends on $Z(s)$ only for $0 \leq s \leq t$. Stopping before time t depends only upon the history up to time t. The event that a stopping time T is less than or equal to t cannot depend on the future of the reference stochastic process $\{Z(s) : s \geq 0\}$ after time t.

Example 2.1 A standard stopping time relative to BM is the **hitting time** of a point a. Let T_a denote the first time t at which $B(t) = a$. Or let $T_{a,b} = \min\{T_a, T_b\}$ be the first time that BM hits either a or b. These hitting times are stopping times.

There is a corresponding definition in discrete time. A nonnegative integer-valued random variable N is a **stopping time** relative to a discrete-time stochastic process $\{X_n : n \geq 0\}$ if, for any time $n \geq 0$, the event $\{N \leq n\}$ depends on $\{X_k : k \geq 1\}$ only for $0 \leq k \leq n$. Stopping before time n depends only upon the history up to time n. The event that a stopping time N is less than or equal to n cannot depend on the future of the reference stochastic process $\{X_k : k \geq 0\}$ after time n.

Example 2.2 Renewal theory provides a nice example in discrete time. Let $\{X_n : n \geq 1\}$ be the successive intervals between renewals and let $\{S_n : n \geq 1\}$ be the associated renewal epochs themselves, where $S_n = X_1 + \cdots + X_n$. Let $\{N(t) : t \geq 0\}$ be the associated renewal counting process. It is important that $N(t)$ is **not** a stopping time with respect to $\{X_n : n \geq 1\}$ or $\{S_n : n \geq 1\}$. The event $N(t) \leq n$ depends on X_{n+1} as well as $X_1, \ldots, X_n$. However, the associated random variable $N(t) + 1$ **is** a stopping time with respect to $\{X_n : n \geq 1\}$ and $\{S_n : n \geq 1\}$.

An important result in renewal theory is **Wald's equation**. Let $\{X_n : n \geq 1\}$ be the successive intervals between renewals, which of course are required to be i.i.d. nonnegative random variables. Wald's equation applies to a stopping time N relative to the sequence $\{X_n : n \geq 1\}$. If N is a stopping time, then

$$E[\sum_{k=1}^{N} X_k] = E[N]E[X_1].$$

Since $N(t) + 1$ is a stopping time relative to $\{X_n : n \geq 1\}$ when $N(t)$ is the renewal counting process, Wald's equation gives us

$$E[\sum_{k=1}^{N(t)+1} X_k] = E[N(t) + 1]E[X_1].$$

This relation is used to prove the **elementary renewal theorem**, Theorem 7.1 in the book. The elementary renewal theorem states a limit for the renewal function $m(t) \equiv E[N(t)]$, in particular,

$$\frac{m(t)}{t} \to \frac{1}{E[X_1]} \quad \text{as} \quad t \to \infty.$$

The proof is outlined in Exercise 7.13. Wald's equation is discussed in other exercises from 7.14 to 7.24.

3 The Optional Stopping Theorem

The **optional stopping theorem** says that, under regularity conditions, when Y is a martingale with respect to Z and T is a stopping time relative to Z, that

$$E[Y(T)] = E[Y(0)] \ .$$

We will apply that below. That also can be expressed by saying that $\{Y(0), Y(T)\}$ is a MG. We remark that the extra regularity conditions are important, because without them the result need not be true, but they will be satisfied in the cases we consider.

4 The Reflection Principle (Section 10.2)

One simple result for hitting times of Brownian motion stems from the **reflection principle**.

For $a > 0$, let T_a be the hitting time of a by BM, i.e., the first time that BM hits a.

$$P(T_a \le t) = P(\max_{0 \le s \le t} B(s) > a) = 2P(B(t) > a) = 2P(N(0,t) > a) = 2P(N(0,1) > a/\sqrt{t}).$$

The second relation above follows from the reflection principle.

5 The Gambler's Ruin Problem

Let $X(t) \equiv \sigma B(t) + \mu t$, for $t \ge 0$. Then X is Brownian motion with drift μ and diffusion or variance coefficient σ^2. Then $X(t) \stackrel{\mathrm{d}}{=} N(\mu t, \sigma^2 t)$ for each $t \ge 0$. Suppose that we start with an initial amount of money w and gamble according to the stochastic process X, i.e., with Brownian motion with drift. Thus our wealth at time t is given by

$$W(t) = w + X(t) = w + \sigma B(t) + \mu t, \quad t \ge 0 \ .$$

We now ask what is the probability we win x (reach wealth $w + x$) before we lose our initial wealth w (reach wealth 0). Let T be the first time that we either win x or lose w, i.e.,

$$T \equiv \inf\{t \ge 0 : W(t) = 0 \quad \text{or} \quad w + x\} = \inf\{t \ge 0 : X(t) = -w \quad \text{or} \quad +x\} \ .$$

We want to find the probability that we win, i.e., reach wealth $w + x$ before being ruined (reaching wealth 0):

$$p \equiv P(W(T) = w + x) = P(X(T) = x)$$

and we want to find the expected time before the "game" ends, i.e., $E[T]$. There are two cases: (i) no drift ($\mu = 0$), and (ii) drift ($\mu \ne 0$). In both cases we can use martingales.

5.1 Three Basic Martingales

We now define three basic martingales with respect to standard Brownian motion B. The **linear martingale** is Brownian motion itself $\{B(t) : t \ge 0\}$. The **quadratic martingale** is $\{B(t)^2 - t : t \ge 0\}$. The **exponential martingale** is $\{e^{\theta B(t) - \theta^2 t/2} : t \ge 0\}$. We find these last two by considering $E[B(t)^2]$ and $E[e^{\theta B(t)}]$ and seeing what deterministic adjustment guarantees that the expectation at t equals the expectation at 0. That does not prove the MG property, but that is the first step. The MG property follows from the stationary and independent increments of BM.

For example, we showed that the quadratic martingale $\{B(t)^2 - t : t \geq 0\}$ is indeed a martingale with respect to Brownian motion. We exploited the independent increments property, writing

$$\begin{aligned}
E[B(t+u)^2 - (t+u)|B(s), \quad 0 \leq s \leq t] &= E[B(t+u)^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\
&= E[(B(t) + B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\
&= E[B(t)^2 + 2B(t)(B(t+u) - B(t)) \\
&\quad +(B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\
&= B(t)^2 + E[2B(t)(B(t+u) - B(t)) \\
&\quad +(B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\
&= B(t)^2 + E[(B(t+u) - B(t))^2|B(s), \quad 0 \leq s \leq t] - (t+u) \\
&= B(t)^2 + E[(B(t+u) - B(t))^2] - (t+u) \\
&= B(t)^2 + E[(B(u) - B(0))^2] - (t+u) \\
&= B(t)^2 + E[B(u)^2] - (t+u) \\
&= B(t)^2 + Var[B(u)] - (t+u) \\
&= B(t)^2 + u - (t+u) \\
&= B(t)^2 - t \ ,
\end{aligned}$$

as claimed.

5.2 The Three Basic Martingales Applied to the Gambler's Ruin Problem

The three basic martingales can be applied with the optional stopping theorem to obtain the desired quantities p and $E[T]$ in the gambler's ruin problem, with and without drift. The appropriate martingale in each case is given in Table 1.

desired quantity	no drift	drift
p	linear martingale	exponential martingale
$E[T]$	quadratic martingale	linear martingale

Table 1: The martingales used to treat the gambler's ruin problem for Brownian motion, with and without drift.

5.3 The Case without Drift

The case without drift is easier. For the Gambler's ruin problem expressed in terms of x and w, we use the linear MG to get

$$p = \frac{w}{x+w}$$

and then we use the quadratic MG to get

$$E[T] = \frac{xw}{\sigma^2} \ .$$

We now explain: We first consider standard Brownian motion, which starts at $B(0) = 0$. For the first step, we let T be the first time that standard Brownian motion first hits either $+a$ or $-b$. We apply the optional sampling theorem to get

$$E[B(T)] = E[B(0)] = 0 \ ,$$

but $B(T)$ necessarily is either $+a$ or $-b$. Let p be the probability that $B(T) = a$. Then the equation is

$$E[B(T)] = pa + (1-p)(-b) = 0 \ ,$$

but that implies that

$$p = \frac{b}{a+b} \ ,$$

which reduces to the formula above when we use $X(t) = \sigma B(t)$, $a = x$ and $b = w$.

By a minor variant of the same reasoning, but with the quadratic martingale, we get

$$E[B(T)^2 - T] = E[B(0)^2 - 0] = 0 \ ,$$

which implies that

$$E[T] = E[B(T)^2] = pa^2 + (1-p)b^2 = \frac{ab(a+b)}{a+b} = ab \ .$$

If we instead consider the stochastic process $X(t) = \sigma B(t)$, then we see that the appropriate quadratic martingale is instead $\{X(t)^2 - \sigma^2 t : t \geq 0\}$. We then get

$$E[X(T)^2 - \sigma^2 T] = E[B(0)^2 - \sigma^2 0] = 0 \ ,$$

which implies that

$$E[\sigma^2 T] = \sigma^2 E[T] = E[X(T)^2] = pa^2 + (1-p)b^2 = \frac{ab(a+b)}{a+b} = ab$$

and

$$E[T] = \frac{ab}{\sigma^2} \ .$$

The final two sections below are optional.

5.4 The Case with Drift

Start by applying the exponential martingale associated with standard Brownian motion $B(t)$ to find p. The first step is to show that the exponential martingale $\{e^{\theta B(t) - \theta^2 t/2} : t \geq 0\}$ is in fact a martingale. Then we apply the Martingale Stopping Theorem (or the optional stopping theorem) to get

$$E[e^{\theta B(T) - \theta^2 T/2}] = E[e^{\theta B(0) - \theta^2 0/2}] = e^0 = 1 \ .$$

Thus, upon expressing $B(t)$ in terms of $X(t)$, we have

$$E[e^{\theta[(X(T) - \mu T)/\sigma] - \theta^2 T/2}] = E[e^{(\theta X(T)/\sigma) - (\theta \mu T/\sigma) - \theta^2 T/2}] = 1 \ .$$

Now let

$$\theta = \frac{-2\mu}{\sigma} \ .$$

That serves to knock out the last two terms in the exponent, giving

$$E[e^{(-2\mu X(T)/\sigma^2)}] = 1 \ .$$

But then use the fact that $X(T) = x$ with probability p and $X(T) = -w$ with probability $1-p$. Thus we get

$$pe^{-2\mu x/\sigma^2} + (1-p)e^{2\mu w/\sigma^2} = 1$$

or

$$p = \frac{e^{+2\mu w/\sigma^2} - 1}{e^{+2\mu w/\sigma^2} - e^{-2\mu x/\sigma^2}} \ .$$

That is the natural way to write the formula when $\mu > 0$; then both the numerator and denominator are positive. You can check that $0 < p < 1$.

If instead $\mu < 0$, then we would write

$$p = \frac{1 - e^{+2\mu w/\sigma^2}}{e^{-2\mu x/\sigma^2} - e^{+2\mu w/\sigma^2}} \ .$$

That makes the numerator and denominator positive.

Now, knowing p, we can apply the linear martingale to find $E[T]$. we have

$$E[B(T)] = 0$$

or, equivalently,

$$E[(X(T) - \mu T)/\sigma] = 0$$

or

$$E[T] = \frac{E[X(T)]}{\mu} = \frac{(px - (1-p)w]}{\mu} \ .$$

We then use the known p and solve for $E[T]$. We get a ratio of linear combinations of exponential terms.

5.5 Other Brownian Martingales

There is also the **cubic martingale**

$$B(t)^3 - 3tB(t), \quad t \geq 0 \ ,$$

and the **quartic martingale**

$$B(t)^4 - 6tB(t)^2 + 3t^2, \quad t \geq 0 \ .$$

For more discussion, see Exercises 10.16-10.24 in Ross. See p. 382 in his more advanced book, *Stochastic Processes* for the case with drift above.