Mathematical Models for Hospital Inpatient Flow Management

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Outline

• Part I: Data
• Part II: Model
• Part III: Analytical analysis
• Part IV: Managerial Insights
Overview

- **Motivation**
  - Inpatient flow management
  - Impact of *early discharge* policy
    - Waiting time for admission to ward
    - Stabilize hourly waiting time performance

- **A stochastic network model**
  - Allocation delays
  - Overflow policy
  - Endogenous service times

- **Predict the time-dependent waiting time**
  - A two-time-scale approach
Part I

- Empirical observations
  - Online Supplement for “Hospital Inpatient Operations: Mathematical Models and Managerial Insights” (68 pages)

- Joint work with
  - James ANG and Mabel CHOU (NUS)
  - Ding DING (UIBE, Beijing)
  - Xin JIN and Joe SIM (NUH)
Capacity and source of admission

- Patients from 4 admission sources competing for inpatient beds

<table>
<thead>
<tr>
<th>Inpatient Beds</th>
<th>General Wards</th>
<th>ED-GW patients</th>
<th>ICU-GW patients</th>
<th>Elective patients</th>
<th>SDA patients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>~600</td>
<td>66.9 (65%)</td>
<td>9.12 (9%)</td>
<td>18.5 (18%)</td>
<td>9.13 (9%)</td>
</tr>
</tbody>
</table>
Key performance measures

• Waiting time for admission to ward (Jan 08 – Jun 09)
  • Waiting time = admission time – bed request time
  • Average: **2.82** hour
  • **6.52%** of ED-GW patients wait more than 6 hours to get a bed
    • 6-hour service level
    • MOH cares

• Quality- and Efficiency-Driven (QED)
  • Average waiting time = **2.3%** (average service time)
  • Average bed utilization = **90%**
Time dependency

- Waiting time depends on patient’s bed request time
- Can we stabilize?

![Avg. waiting time](image)

![6-h service level](image)
Time-varying bed request rate

- ED-GW patient’s bed request rate (red curve) depends on arrival rate to ED (blue curve)
Learning from call center research?

  - Staffing of Time-Varying Queues to Achieve Time-Stable Performance

- Yunan Liu and Ward Whitt, 2012
  - Stabilizing customer abandonment in many-server queues with time-varying arrivals
Mismatch between demand and supply of beds

- Jan 08 – Jun 09
Discharge policy

• Discharge timing affects the waiting time

• Early discharge policy
  • Moving the discharge time a few hours earlier in the day

• The hospital implemented early discharge policy since July 2009
  • Study two periods of data
    • Jan 2008 to Jun 2009 (Period 1)
      • 13% before noon
    • Jan 2010 to Dec 2010 (Period 2)
      • 26% before noon
Waiting time for ED-GW patients

<table>
<thead>
<tr>
<th></th>
<th>1\textsuperscript{st} period</th>
<th>2\textsuperscript{nd} period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average waiting time</td>
<td>2.82 h</td>
<td>2.77 h</td>
</tr>
<tr>
<td>6-hour service level</td>
<td>6.52%</td>
<td>5.13%</td>
</tr>
</tbody>
</table>
Challenges

• Does the modest improvement come from the early discharge?
  • Changing operating environment
  • Both arrival volume and capacity increases during 2008 to 2010
  • Bed occupancy rate (BOR) reduces in the Period 2
    • Period 1: 90.3%
    • Period 2: 87.6%

• More importantly, is there any operational policy that can stabilize the waiting time?

• Need a model to help
Part II: A stochastic model

- Model
  - Hospital Inpatient Operations: Mathematical Models and Managerial Insights, submitted

- Joint work with Mabel Chou, Ding Ding, and Joe Sim
A multiclass, multi-server pool system

Diagram showing a system with multiple servers and buffers. The servers are labeled as ED-GW, ICU-GW, and SDA. Each server has 9 buffers. The servers are connected to various patient categories such as Neuro, Renal, Gastro, Surg, Card, Ortho, Onco, Gen-Med Respi, Gen-Med Neuro, Surg Ortho, Surg Card, Respi Surg, Overflow I, Overflow II, and Overflow III.
Time-varying arrival rates
Specialty distribution
Key modeling components

- Service time model
  - Determined by admission time, **LOS** and discharge distribution
  - An endogenous modeling element
  - No longer i.i.d.

- Allocation delays
  - “Secondary” bottlenecks other than bed availability
    - Yankovic and Green (2011)
    - Armony et al (2011)

- Overflow policy
  - When to overflow a patient
  - Overflow to which server pool
Simulation replicates most performance measures

- Hourly waiting time performances

(a) Hourly average waiting time

(b) Hourly 6-hour service level
Time-dependent queue length
Service times are endogenous

- Service time model
  - Service time = Discharge time – Admission time
    \[= \text{LOS} + \text{Dis hour} - \text{Adm hour}\]

- LOS distribution
  - Average is \(~ 5\) days
  - Depend on admission source and specialty
  - AM- and PM- dependent for ED-GW patients

Length of Stay (LOS)
\[= \text{Discharge day} - \text{Adm day}\]
Verify the service time model

- Service time model
  - Service time = LOS + Discharge hour – Adm hour

Matching empirical

(a) Empirical (Armony et al 2011)  
(b) Simulation output

![Histogram of service time](image1)

![Histogram of service time](image2)
Pre- and post-allocation delays

- Patient experiences additional delays upon arrival and when a bed is allocated
  - Pre-allocation delay
    - BMU search/negotiate for beds
  - Post-allocation delay
    - Delays in ED discharge
    - Delays in the transportation
    - Delays in ward admission

- Must model allocation delays
  - If not, hourly queue length does not match (right figure)
Time-dependent allocation delays

- The mean of allocation delay depends on when it is initiated
  - Use log-normal distribution
  - Pre-allocation delay
Overflow policy

- When a patient’s waiting time exceeds certain threshold, the patient can be overflowed to a “wrong” ward
- Beds are partially flexible
- Overflow wards have certain priority

<table>
<thead>
<tr>
<th>Cluster</th>
<th>1st Overflow</th>
<th>2nd Overflow</th>
<th>3rd Overflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medicine</td>
<td>Other Med</td>
<td>Surgery / OG</td>
<td>Ortho</td>
</tr>
<tr>
<td>Surgery</td>
<td>Other Surg</td>
<td>Ortho / OG</td>
<td>Medicine</td>
</tr>
<tr>
<td>Ortho</td>
<td>Other Ortho</td>
<td>Surgery</td>
<td>Medicine</td>
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</table>
Dynamic overflow policy

**Fixed threshold**
- Threshold: 4.0 h

**Dynamic threshold**
- Threshold: 0.5 h for arrival between 7 pm and 7 am (next day); 5.0 h for others
Part III: Analytical analysis

- Two-time scale method to predict time-dependent performance measures
Two-time scale

- Discrete queue
  - Average LOS and daily arrival rate determine $\{X_k\}$, and thus performances at mid-night (daily level)

- Time-varying performance
  - The arrival rate pattern, discharge timing, and allocation delay distribution determine the hour-of-day behavior
A simplified model

- Single cluster
  - No overflow
- Arrival is periodic Poisson
- LOS follows a Geometric distribution
- Discharge follows a simple discrete distribution

- Service time follows the non-iid model:
  \[ S = \text{LOS} + h_{\text{dis}} - h_{\text{alloc}} \]
  - Admission time is replaced by allocation time

- Allocation delay
  - Each customer experiences a random delay after allocation time
Predict the time-dependent average queue length

- Decompose the queue length into two parts
  - Queue for beds: patients who are waiting for a bed
  - Alloc-delay queue: patients who are allocated with beds and are experiencing the alloc-delay
Queue for bed (1/2)

- \( X_k \) denotes the number of customers at midnight of day \( k \)
  
  \[
  X_{k+1} = X_k - \Phi_k + \Lambda_k
  \]

- Discrete queue

- Number of discharges only depends on \( X_k \) since
  - LOS is geometric (“coin toss” every day)
  - LOS starts from 1 (i.e., no same-day discharge)

- Number of arrivals follows Poisson distribution
  - Independent of number of discharges

- \( \{X_k\} \) is a Markov process
  - Stationary distribution can be solved explicitly
  - Ramakrishnan et al. (2005)
Queue for bed (2/2)

- Using the stationary distribution of \( \{X_k\} \)
  - The average number of customers in system and the average queue length can be obtained for any time point
  - Average number of customer in system can be solved in a fluid way
    - \( E[Y(t)] = E[X_k] + \int_0^t \lambda(t)dt - E[\text{discharge}(0, t)] \)
    - Powell et al. (2012)

- Queue length needs to be obtained from the distribution of number of customers in system at each time point \( Y(t) \)
  - Conditioning on \( X_k \)
    - \( Y(t) \) is a convolution between arrival (Poisson r.v.) and discharge (Binomial r.v. depends on the value of \( X_k \)) till \( t \)
Related work


  - The relationship between inpatient discharge timing and emergency department boarding

- Affiliations: Department of Emergency Medicine, Northwestern University; Harvard Affiliated Emergency Medicine Residency, Brigham and Women’s Hospital–Massachusetts General Hospital, ...
Alloc-delay queue

- Each patient experiences a random amount of delay
  - The alloc-delays follow an iid distribution with CDF $F(x)$
  - Patient gets a bed before entering the alloc-delay queue

- Two scenarios
  - Unlucky patient: no bed available upon arrival
    - Waits in the queue for bed first
    - Gets a bed at a discharge time point
  - Lucky patient: gets a bed allocated upon arrival
    - Directly joins the alloc-delay queue
Unlucky patients

• Suppose discharges occur at $t_1, t_2, t_3, t_4$
• The mean number of admissions at each discharge point can be calculated from $X_k$, arrivals and discharges

• Given the mean number of admissions $Z(t_i)$
  • Mean number of customers in the alloc-delay queue after $s$ hours is $Z(t_i) \cdot (1 - F(s))$
Lucky patients

• The *effective* admission process (bed-allocation process) is non-homogeneous Poisson
  • The probability of an arriving patient being lucky or unlucky is independent of the arrival itself
  • The effective admission rate can be calculated from $X_k$, arrivals and discharges

• Consider the alloc-delay queue as an infinite-server queue
  • Service time is the allocation delay
  • The effective admission process constitutes the arrival
  • Infinite-server queue theory (Eick - 1993):
    \[
    m(t) = E[\lambda(t - F_e)]E[F]
    \]
Numerical results

- Alloc delays follow iid exponential distribution with mean 2 hours
- Simple discrete distribution:
Numerical results

- Avg queue length

![Graph showing average queue length over time for different peak times: 11am, 1pm, and 3pm. The graph indicates fluctuations in queue length with peak times corresponding to the highest points.](image-url)
Insights from the simplified model

- The average number of customers in the system remain the same in scenarios with and without allocation delays.

- Challenging to predicting the hourly queue length
  - Necessary to model allocation delays
    - Slower drop in the queue length after 2pm

- Early discharge helps stabilize the hourly queue length.
Shift the Period 1 discharge curve

- Using constant-mean allocation delay
- Avg queue length

Avg waiting time
Part IV: Managerial insights

- Whether early discharge policy is beneficial or not
- What-if analysis
Simulation results

- Simulation shows NUH early discharge policy has little improvement
  (a) hourly avg. waiting time
  (b) 6-hour service level
Aggressive early discharge policy
Aggressive early discharge + smooth allocation delay

- Waiting time performances can be stabilized
  (a) hourly avg. waiting time
  (b) 6-hour service level
Only use aggressive early discharge

- Cannot be stabilized
  - (a) hourly avg. waiting time
  - (b) 6-hour service level
Only smooth the allocation delays

- Assuming allocation delay has a constant mean
  (a) hourly avg. waiting time
  (b) 6-hour service level
Impact of capacity increase

- 10% reduction in utilization, plus assuming allocation delay has a constant mean
  (a) hourly avg. waiting time
  (b) 6-hour service level
Summary

• Conduct an empirical study of patient flow of the entire inpatient department

• Build and calibrate a stochastic model to evaluate the impact of discharge distribution on waiting for admission to ward

• Analyze a simplified version of the stochastic model using a two-time scale approach

• Achieve stable waiting time by aggressive early discharge + smooth allocation delay
Questions?
Limitations

- Simulation cannot fully calibrate with the overflow rate
  - Bed class (A, B, C)
  - Gender mismatch
  - Hospital acquired infections
    - Example: a female Surg patient has to be overflowed to a Med ward, since the only available Surg beds are for males

- Day-of-week phenomenon
  - Admission and discharge both depends on the day of week
  - LOS depends on admission day
  - Performances (BOR, waiting time) varies among days
Simulation replicates most performance measures

- Hourly waiting time performances

(a) Hourly average waiting time
(b) Hourly 6-hour service level
Average queue length (simulation result)
Average waiting time for each specialty

- Renal patients have longest average waiting time
6-hour service level for each specialty

- Cardio and Oncology patients show significant improvement in the 6-hour service level
Overflow rate

- Overall overflow rate reduces in Period 2
Background

- One of the major hospitals in Singapore
  - Around 1,000 beds in total

- 38 inpatient wards
  - We focus on 21 general wards
  - ICU, ISO, pediatric wards are excluded
  - Wards are dedicated to one specialty or shared by two and more specialties

- Serving around 90,000 patients annually
  - Data from 2008 to 2010
Time dependency

- Waiting time depends on patient’s bed request time
- Use time exit from ED
- Jan 08 – Jun 09
Waiting time for ED patients (using MOH definition)

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<tr>
<td>Average waiting time</td>
<td>2.50 h</td>
<td>2.44 h</td>
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<tr>
<td>6-hour service level</td>
<td>5.24%</td>
<td>3.90%</td>
</tr>
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</table>

- (a) hourly avg. waiting time
- (b) hourly 6-hour service level
Histogram of waiting time (MOH definition)
Histogram of service time

- Resolution of 1 hour
- Period 1
- Period 2
Log-normal fit for LOS distribution
Relation between residual, $T_{\text{adm}}$, and $T_{\text{dis}}$

- Residual

\[
\text{res}(S) = S - \lfloor S \rfloor = T_{\text{dis}} - T_{\text{adm}} - \lfloor (T_{\text{dis}} - T_{\text{adm}}) \rfloor = (T_{\text{dis}} - \lfloor T_{\text{dis}} \rfloor - (T_{\text{adm}} - \lfloor T_{\text{adm}} \rfloor)) \mod 1,
\]

where for two real numbers $x$ and $y \neq 0$, $x \mod y = x - \lfloor x/y \rfloor \cdot y$. 
Alternative service time model (1/2)

- $S = T_{\text{dis}} - T_{\text{adm}}$
  - $S$ denote service time (in unit of day)
  - $T_{\text{adm}}$ denote the admission time, $T_{\text{dis}}$ denote the discharge time

- Residual = $S - \text{floor}(S)$
  - histogram (right fig)

- In the alternative model
  - Generate the integer part $\text{floor}(S)$ from empirical distribution
  - *Independently* generate the residual from another empirical distribution
Alternative service time model (2/2)

- Histogram of residual conditioning on each integer value
- The conditional distribution are close, except when \( \text{floor}(S) = 0 \)
Alternative service time model

- If directly generating service time
  - Discharge distribution does not match
  - Avg. waiting time does not match
Stochastic network models

- Multiclass, multi-server pools with some flexible pools
  - 30 ~ 60 servers in each pool
  - 15 server pools

- Typical BOR is 86% ~ 93%

- Periodic arrival processes

- Long service times = several arrival periods
  - Average LOS = 5 days

- Waiting time is a small fraction of service time
  - Average waiting time = 2.5 hours = 1/48 average LOS

- Must overflow in a fraction of the service time
Simulation model

- Using 9 cluster of patients and 15 server pools
  - **Utilization (Sim):** 90.5%; **(empirical):** 88.0%
  - We did not catch gender/ bed class /sub-specialty mismatch in simulation

- 4 types of arrivals for each cluster
  - ED-GW
  - EL
  - ICU-GW
  - SDA
    - Use empirical arrival rate and service time for each type of patients
Analytical results: no allocation delay

- Compare with simulation results
  - Number of customer in system
  - Avg queue length

![Graph showing number of customers and average queue length over time with peaks at 11am, 1pm, and 3pm.]
A stochastic model

- Multi-class, multi-server pool system
  - Each server pool is either dedicated to one class of customer or flexible to serve two and more classes of customers

- Periodic arrival
  - 4 types of arrival (ED-GW, Elective, ICU-GW, SDA) for each specialty

- A novel service time model

- And other key components
The admission time affects LOS
- AM patients: average LOS = 4.24 days
- PM patients: average LOS = 5.31 days