Experience Statistical Regularity

By Looking at Random Walks
Simulation Experiments
Plotting Random Walks

$X_1, X_2, \ldots$ IID random variables

$S_n = X_1 + \cdots + X_n, \quad n \geq 1,$

with $S_0 = 0$ partial sums

Plot $S_0, S_1, \ldots, S_n$

To start: $X_i = U_i$ uniformly distributed on $[0, 1]$. 
What should we see?
Look at larger sample sizes!
What should we see?

Looking at $S_1, \ldots, S_n$
when $n = 10^j$ for $j = 1, 2, 3, 4$
Plots for \( n = 10^j \) with \( j = 1, 2, 3, 4 \)
How does plotting work?
Map Into the Unit Square

(if we ignore the units on the axes)
Fit horizontally: create a function on $[0, 1]$.

For $y_0, y_1, \ldots, y_n$ given, let $x : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$x(t) \equiv y_{\lfloor nt \rfloor}, \quad 0 \leq t \leq 1,$$

where $\lfloor z \rfloor$ is the greatest integer less than or equal to $z$. 
The Plot Function: Step 2

Fit vertically.

Place between infimum and supremum.
For $x : [0, 1] \rightarrow \mathbb{R}$ given,

$$\text{plot}(x) \equiv \left( x - \inf(x) \right) / \text{range}(x),$$

where

$$\inf(x) \equiv \inf\{x(t) : 0 \leq t \leq 1\}$$

$$\sup(x) \equiv \sup\{x(t) : 0 \leq t \leq 1\}$$

$$\text{range}(x) \equiv \sup(x) - \inf(x)$$
When you plot a random walk,

You get a random plot.

You get a random function.

a random function mapping $[0,1]$ into $[0,1]$.

You get a stochastic process.
Plots for $n = 10^j$ with $j = 1, 2, 3, 4$
Modified Experiment

Let \( X_i = U_i - 0.5. \)

Construct centered random walk.

What should we see now?
The centered random walk for $n = 10^j$ with $j = 1, 2, 3, 4$
six cases for $n = 10^4$
six cases for $n = 10^6$
You see Brownian motion!!!
Density Estimates

for last partial sum $S_n$
density estimate for $S_n - mn$ with uniformly distributed summands
density estimates for \( S_n - mn \) with uniformly distributed summands
The Random Plot Limit

**Theorem.** If the stochastic-process limit $S_n \Rightarrow S$ holds, where

$$S_n(t) \equiv (S_{\lfloor nt \rfloor} - m\lfloor nt \rfloor)/c_n, \quad 0 \leq t \leq 1,$$

for some constants $m$ and $c_n : n \geq 1$, and

$$P(\text{range}(S) = 0) = 0,$$

then

$$\text{plot}(S_k - mk : 0 \leq k \leq n) \Rightarrow \text{plot}(S).$$
Invariance Principles
New Random Steps

Let \( Y_i = f(U_i) \).

\( X_i = Y_i - EY_i \).

new centered random walk.

What should we see now?
Three Cases

(i) $f(U) = -m \log(U)$ for $m = 1, 10$

(ii) $f(U) = U^p$ for $p = 1/2, 3/2$

(iii) $f(U) = U^{-1/p}$ for $p = 1/2, 3/2$
Cases (i) and (ii) for $n = 10^4$
The Exception Makes the Rule
Case (iii)

(i) \( f(U) = -m \log(U) \) for \( m = 1, 10 \)

(ii) \( f(U) = U^p \) for \( p = 1/2, 3/2 \)

(iii) \( f(U) = U^{-1/p} \) for \( p = 1/2, 3/2 \)
Plots of the uncentered random walk for $U^{-1/p}$ with $p = 3/2$
Plots of the uncentered random walk for $U^{-1/p}$ with $p = 1/2$
$P(U^{-1/p} > t) = P(U^{1/p} < t^{-1})$. 

$= P(U < t^{-p}) = t^{-p}$

Has infinite mean for $0 < p \leq 1$

Has infinite variance for $0 < p \leq 2$
Plots of the Centered Random Walk

for $U^{-1/p}$ with $p = 3/2$
Plots of the centered random walk for $U^{-1/p}$ with $p = 3/2$
six cases for $n = 10^4$
six cases for $n = 10^6$
More Plots of the Uncentered Random Walk

for $U^{-1/p}$ with $p = 1/2$
six cases for $n = 10^4$
six cases for \( n = 10^6 \)
Conclusions

Plotting reveals statistical regularity.

Stochastic-process limits explain the statistical regularity.