The LLN, CLT and Extensions

Used for Confidence Intervals and HT Approximations for Queues

IEOR 4615, Service Engineering, Professor Whitt

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OUTLINE

1. The Classical Limit Theorems of Probability Theory
   - Law of Large Numbers (**LLN**)
   - Central Limit Theorem (**CLT**)
2. Corresponding Limits for Counting (e.g., Arrival) Processes
3. Functional Versions: Limits for Stochastic Processes
   - Functional Law of Large Numbers (**FLLN**)
   - Functional Central Limit Theorem (**FCLT**)
4. Corresponding Limits for Counting (e.g., Arrival) Processes
5. Heavy-Traffic (HT) Limits for Queueing Models

The Classical Limit Theorems of Probability Theory

Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables distributed as X and let S_n be the associated partial sum $S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$ with $S_0 \equiv 0$. (§2.8 of Ross (2010))

Strong LLN: If $E[|X|] < \infty$, then

$$\bar{X}_n \equiv \frac{S_n}{n} \to E[X] \quad \text{as} \quad n \to \infty \quad \text{with probability 1 (w.p.1)}$$

CLT: If $Var(X) \equiv \sigma^2 < \infty$, then

$$\sqrt{n}(\bar{X}_n - E[X]) \equiv \frac{1}{\sqrt{n}}(S_n - nE[X]) \Rightarrow N(0, \sigma^2)) \quad \text{as} \quad n \to \infty.$$

What Does the CLT Say and How Is it Used?

1. What does the CLT say? $\Rightarrow$ means convergence in distribution:

$$P\left(\frac{1}{\sqrt{n}}\left(S_n - nE[X]\right) \le x\right) \to P\left(N(0,\sigma^2) \le x\right) \quad \text{as} \quad n \to \infty$$

for all x, where $N(0,\sigma^2) \stackrel{\mathrm{d}}{=} \sigma N(0,1)$ and $N(0,1)$ is a standard (mean 0 and variance 1) normal random variable, with

$$P(N(0,1) \le x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2}\, dy$$

2. How is the CLT Used? **Normal Approximation:** To approximate the probability distribution of S_n:

$$S_n \approx N(nE[X], n\sigma^2) \stackrel{\mathrm{d}}{=} nE[X] + \sqrt{n\sigma^2}N(0,1)$$

Normal Approximation

$$
\begin{aligned}
P(S_n < x) &= P(S_n - nE[X] < x - nE[X]) \\
&= P\left(\frac{S_n - nE[X]}{\sqrt{Var(X)}} < \frac{x - nE[X]}{\sqrt{Var(X)}}\right) \\
&\approx P\left(N(0,1) < \frac{x - nE[X]}{\sqrt{Var(X)}}\right)
\end{aligned}
$$

Look up in table.

Application: Confidence Interval for Mean Service Time

1. Sample Mean of Service times: Given service times $X_1, X_2, \ldots, X_n$,

$$\bar{X}_n \equiv \frac{X_1 + \cdots + X_n}{n}$$

2. Apply CLT: Assuming i.i.d. with mean $E[X_k] = m$ and $Var(X_k) = \sigma^2$,

$$\bar{X}_n \approx N(m, \sigma^2/n)$$

3. Basis for 95% confidence interval estimate for true mean m:

$$[\bar{X}_n - \frac{z_{0.025}\sigma}{\sqrt{n}}, \bar{X}_n + \frac{z_{0.025}\sigma}{\sqrt{n}}]$$

if variance known; otherwise use estimate of variance and the Student-t distribution.

Associated Limits for Renewal Counting Processes

Let $X_1, X_2, \ldots$ be i.i.d. nonnegative variables distributed as X and let

$S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$ with $S_0 \equiv 0$ and counting process

$A(t) \equiv \max\{k \geq 0 : S_k \leq t\}$ (renewal process, §7.3 of Ross (2010))

Strong LLN: If $E[|X|] < \infty$, then

$$\frac{A(t)}{t} \to \frac{1}{E[X]} \quad \text{as} \quad t \to \infty \quad \text{with probability 1 (w.p.1)}$$

CLT: If $Var(X) \equiv \sigma^2 < \infty$, then

$$\frac{1}{\sqrt{t}}\left(A(t) - (t/E[X])\right) \Rightarrow N(0, \sigma^2/E[X]^3) \quad \text{as} \quad n \to \infty.$$

Application: Confidence Interval for Arrival Rate

1. Estimate of arrival rate λ: Given arrival counting process $A(t)$ over interval $[0, t]$, the estimated rate is

$$\bar{\lambda}(t) \equiv \frac{A(t)}{t}.$$

2. Apply CLT: Assuming renewal with mean $E[X] = 1/\lambda$ and variance σ^2,

$$\bar{\lambda}(t) \approx N(\lambda, \lambda^3\sigma^2/t)$$

3. Basis for 95% confidence interval estimate for true rate λ:

$$[\bar{\lambda}(t) - \frac{z_{0.025}[\bar{\lambda}(t)]^{3/2}\sigma}{\sqrt{t}}, \bar{\lambda}(t) + \frac{z_{0.025}[\bar{\lambda}(t)]^{3/2}\sigma}{\sqrt{t}}]$$

if variance known; else use estimate of variance and Student-t dist.

Functional Versions: FLLN and FCLT

1. As before, let $X_1, X_2, \ldots$ be i.i.d. variables distributed as X and let $S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$ with $S_0 \equiv 0$.
2. Limit for **all** the partial sums, S_k, $0 \leq k \leq n$, instead of just the last partial sum S_n (see Ch. 1 of WW (2002) book on line)
3. Put in a **common framework: the space D** of right-continuous real-valued functions on $[0, 1]$ with left limits
4. $\mathbf{S_n(t)} \equiv S_{\lfloor nt \rfloor}$, $0 \leq t \leq 1$, where $\lfloor x \rfloor$ is the **floor function**, the greatest integer less than or equal to x.

Functional Versions: FLLN and FCLT

Let $X_1, X_2, \ldots$ be i.i.d. random variables distributed as X and let

$S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$ with $S_0 \equiv 0$.

FLLN: If $E[|X|] < \infty$, then $\bar{\mathbf{S}}_\mathbf{n}(\mathbf{t}) \equiv \frac{S_{\lfloor nt \rfloor}}{n} \to E[X]t$ as $n \to \infty$ w.p.1.

FCLT: If $Var(X) \equiv \sigma^2 < \infty$, then

$$\hat{\mathbf{S}}_\mathbf{n}(\mathbf{t}) \equiv \frac{1}{\sqrt{n}} \left(S_{\lfloor nt \rfloor} - ntE[X] \right) \Rightarrow \sigma \mathbf{B}(t) \quad \text{as} \quad n \to \infty.$$

where $\{\mathbf{B}(t) : t \geq 0\}$ is standard **Brownian motion**. Convergence of stochastic processes (functions of t) as $n \to \infty$.

Continuous Mapping Theorem (CMT)

If $Z_n \Rightarrow Z$ in $\mathcal{S}_1$ as $n \to \infty$ and if $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a continuous function, then

$$f(Z_n) \Rightarrow f(Z) \quad \text{in} \quad \mathcal{S}_2 \quad \text{as} \quad n \to \infty.$$

More useful in a space of functions than for real-valued random variables; e.g., we can apply the CMT with FLLN and FCLT to deduce FLLN and FCLT, and thus LLN and CLT, for queueing models (heavy-traffic limits). (See PP. 24 , 84 of WW book)

In particular, we can apply with **reflection map** $r : D \to D$, with

$$r(x)(t) \equiv x(t) - \inf_{0 \le s \le t} x(s), \quad 0 \le t \le 1.$$

Associated Functional Versions for Counting Processes

In the setting above with same conditions, (§7.3 of WW book (2002))

FLLN: $\bar{A}_n(t) \equiv \frac{A(nt)}{n} \to \frac{t}{E[X]}$ as $n \to \infty$ w.p.1.

FCLT:

$$\hat{A}_n(t) \equiv \frac{1}{\sqrt{n}} \left(A(nt) - \frac{nt}{E[X]} \right) \Rightarrow \sqrt{\sigma^2/E[X]^3}\mathbf{B}(t) \quad \text{as} \quad n \to \infty.$$

where $\{\mathbf{B}(t) : t \geq 0\}$ is standard **Brownian motion**. Convergence of stochastic processes (functions of t) as $n \to \infty$.

$$\text{Note: } \sigma^2/E[X]^3 = (1/E[X]) \times (\sigma^2/E[X]^2) = \lambda c_X^2.$$

Review from Lecture 11, February 26

Application of FCLT and CMT to establish conventional HT limits for $M/M/1/\infty$ and $GI/GI/s/\infty$ queueing models.

HT Limit for $M/M/1/\infty$ Queue Length Process ($\rho = 1$)

- As $n \to \infty$,
- arrival process: $[A(nt) - \lambda nt]/\sqrt{n} \Rightarrow \sqrt{\lambda}\mathbf{B}_a(t)$ (BM limit)
- potential service process: $[S(nt) - \mu nt]/\sqrt{n} \Rightarrow \sqrt{\mu}\mathbf{B}_s(t)$
- If $\lambda = \mu$ or, equivalently, if $\rho = 1$, then
  - net input process: $\frac{X(nt)}{\sqrt{n}} \equiv \frac{A(nt)-S(nt)}{\sqrt{n}}$
    $\Rightarrow \sqrt{\lambda}\mathbf{B}_a(t) - \sqrt{\lambda}\mathbf{B}_s(t) \stackrel{\mathrm{d}}{=} \sqrt{2\lambda}\mathbf{B}(t)$. (again BM limit)
  - queue length process: $\frac{Q(nt)}{\sqrt{n}} \Rightarrow \mathbf{Q}(t) \equiv \mathbf{X}(t) - \inf_{0 \le s \le t}\{\mathbf{X}(s)\}$
  - where $\mathbf{X}(t) \equiv \sqrt{2\lambda}\mathbf{B}(t)$.
  - $\mathbf{Q}(t)$ is **reflected Brownian motion (RBM)**.

HT Limit for $M/M/1/\infty$ Queue Length Process with Drift

- As $n \to \infty$, with λ_n function of n,
- If $(\lambda_n - \mu)\sqrt{n} \to c$, i.e., if $\rho_n \equiv 1 - (c/\sqrt{n})$, then
- arrival process: $[A_n(nt) - \lambda_n nt]/\sqrt{n} \Rightarrow \sqrt{\mu}\mathbf{B}_a(t)$ (BM limit)
- potential service process: $[S(nt) - \mu nt]/\sqrt{n} \Rightarrow \sqrt{\mu}\mathbf{B}_s(t)$
  - net input process: $\frac{X_n(nt)}{\sqrt{n}} \equiv \frac{A_n(nt) - S(nt) - (\lambda_n - \mu)nt}{\sqrt{n}}$

    $$\Rightarrow \sqrt{\mu}\mathbf{B}_a(t) - \sqrt{\mu}\mathbf{B}_s(t) - ct \stackrel{\mathrm{d}}{=} \sqrt{2\mu}\mathbf{B}(t) - ct. \quad \text{(BM with drift)}$$
  - queue length process: $\frac{Q(nt)}{\sqrt{n}} \Rightarrow \mathbf{Q}(t) \equiv \mathbf{X}(t) - \inf_{0 \le s \le t}\{\mathbf{X}(s)\}$
  - where $\mathbf{X}(t) \equiv \sqrt{2\mu}\mathbf{B}(t) - ct$.
  - $\mathbf{Q}(t)$ is reflected Brownian motion (RBM) with drift.
  - The steady-state distribution of RBM with drift is **exponential**!

HT Limit for $GI/GI/s/\infty$ Queue Length Process with Drift

- As $n \to \infty$, **variation of same reasoning applies:**
- If $(\lambda_n - \mu)\sqrt{n} \to c$, then
- arrival process: $[A_n(nt) - \lambda_n nt]/\sqrt{n} \Rightarrow \sqrt{\mu c_a^2}\mathbf{B}_a(t)$ (BM limit)
- potential service process: $[S(nt) - \mu nt]/\sqrt{n} \Rightarrow \sqrt{\mu c_s^2}\mathbf{B}_s(t)$
  - net input process: $\frac{X_n(nt)}{\sqrt{n}} \equiv \frac{A_n(nt) - S(nt) - (\lambda_n - \mu)nt}{\sqrt{n}}$

    $\Rightarrow \sqrt{\mu c_a^2}\mathbf{B}_a(t) - \sqrt{\mu c_s^2}\mathbf{B}_s(t) - ct \stackrel{\mathrm{d}}{=} \sqrt{\mu(c_a^2 + c_s^2)}\mathbf{B}(t) - ct.$
  - queue length process: $\frac{Q(nt)}{\sqrt{n}} \Rightarrow \mathbf{Q}(t) \equiv \mathbf{X}(t) - \inf_{0 \le s \le t}\{\mathbf{X}(s)\}$
  - where $\mathbf{X}(t) \equiv \sqrt{\mu(c_a^2 + c_s^2)}\mathbf{B}(t) - ct.$
  - $\mathbf{Q}(t)$ is reflected Brownian motion (RBM) with drift.
  - The steady-state distribution is **again exponential, but with** $(c_a^2 + c_s^2)$!

References

Background on HT Limits and Approximations

- W^2. ***Stochastic-Process Limits,*** Springer, 2002: http://www.columbia.edu/∼ww2040/book.html (See Chapters 1, 2, 5 and 9 plus §7.3.
- W^2. **The Queueing Network Analyzer.** Bell System Technical Journal 62 (9) (1983) 2779-2815. See §5.1 and §5.2.