Statistical Analysis of Arrival Data

From Service System Data to Arrival Process Models

IEOR 4615, Service Engineering, Professor Whitt

Lecture 17, April 7, 2015

Toward a Daily Arrival Process Model

- What do we **anticipate**?
  - We anticipate a **Nonhomogeneous Poisson Process (NHPP)**.
  - For staffing, we may want **piecewise-constant arrival rate function**.
  - Problem with multiple days: **day-to-day variation** **(over-dispersion)**.
- To confirm, we perform **statistical tests** (**could be whole course**).
  - Today: **On Kolmogorov-Smirnov Tests of NHPP**.
  - Based on **2005 paper by Brown et al.** and two **2014 papers by Song-Hee Kim and WW** plus **recent 2014 paper**
- Given NHPP, we only need to **estimate the arrival rate function**.
  - Using historical data: **forecasting** (Friday). (**could be whole course**).

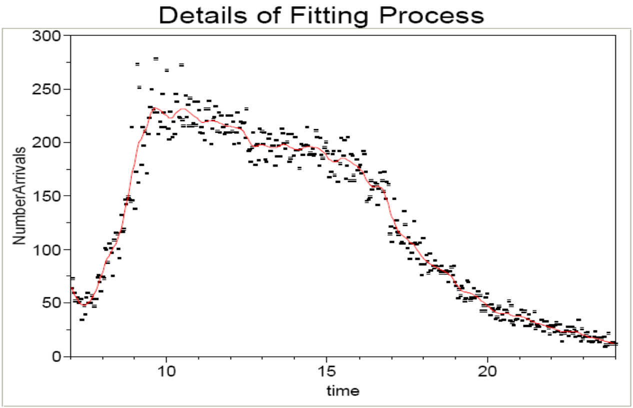
Important Concepts Covered Today, I

1. statistical hypothesis testing
   - null hypothesis (H_0)
   - significance level
   - p-value
   - alternative hypothesis (H_1)
   - power of a statistical test ($P(\text{reject}\, H_0 | H_1)$)
2. Comparing Cumulative Distribution Functions (CDF's)
   - Q-Q plot
   - empirical cdf
   - Kolmogorov-Smirnov (KS) statistical test

Important Concepts Covered Today, II

1. statistical test of a Poisson process
    - the standard KS test (use iid exponential interarrival times)
    - conditional-uniform property of the Poisson process
    - CU KS test
2. statistical tests of an NHPP
    - CU KS test
    - the logarithm test from Brown (2005)
    - the Lewis (1965) test exploiting Durbin (1961) used in KW (2014a)
    - over-dispersion (relative to Poisson process), KW (2014b)

Identifying the Predictable and Unpredictable Variability

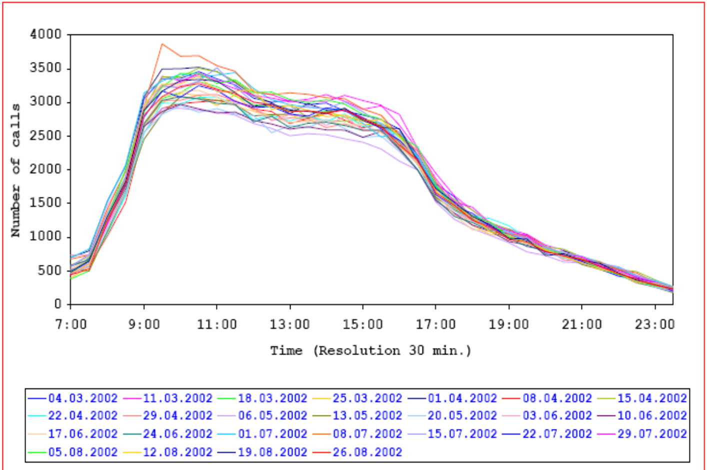


Plot for Aug 9 (Fri.)

- Divide day (7am to midnight) into time intervals of 150 seconds (=2½ minutes)
- Count number arrivals in each interval, and make scatterplot
- Fit using a nonparametric regression smoothing

Look at Multiple Days: IID NHPP's?

Number of Calls at a U.S. bank.
Mondays. March 2002-August 2002.



Coping with Day-to-Day Variation (Over-Dispersion)

1. We address over-dispersion directly through daily totals.
   - We ask if daily totals are consistent with Poisson.
   - Is the variance equal to the mean?
   - We estimate the distribution of the daily totals.
2. We avoid the issue in the test of an NHPP.
   - We do so by using the conditional-uniform property of a PP.
     - (to be explained in following slides)
   - An NHPP can pass the test even if there is over-dispersion.
   - We thus test the NHPP property conditional on the daily total.

Statistical Tests of a NHPP

1. **Reduce to statistical tests of a Poisson Process (PP).**
   - Assume **piecewise-constant arrival rate function**.
   - Then **independent PP's over subintervals**.
2. **Interarrival times iid exponential on each interval, but we would need to estimate mean of each, so we do not use that approach.**
3. **Exploit Conditional Uniform (CU) Property of PP. (first key idea)**
   - n arrival times A_k in [0,T]: A_k/T **are n ordered iid uniforms on [0,1]**.
   - No nuisance parameter: independent of arrival rate.
   - We can combine data from different intervals and days.
   - Use **Kolmogorov-Smirnov test**. (To be discussed in following slides.)

Conditional-Uniform (CU) Property of a PP

- Theorem. Given n arrival times A_k of a PP in [0,T], **A_k/T are distributed as order statistics of n iid uniform variables on $[0,1]$**.

- Proof. For $0 < t_1 < t_2 < \cdots < t_n < T$,

$$
\begin{aligned}
& f_{A_1,\ldots,A_n|A(T)}(t_1,\ldots,t_n|n) \\
& \approx P(N(t_i+\delta)-N(t_i)=1, 1\le i\le n, \text{no other points}|N(T)=n) \\
& \approx \frac{e^{-\lambda t_1}(\lambda\delta e^{-\lambda\delta})e^{-\lambda(t_2-t_1)}(\lambda\delta e^{-\lambda\delta})\cdots e^{-\lambda(T-t_n)}}{\delta^n e^{-\lambda T}(\lambda T)^n/n!} \\
& \to \frac{n!}{T^n} \quad \text{as} \quad \delta\downarrow 0.
\end{aligned}
$$

- (Limiting form of n-dimensional pdf. See §5.3.5 of Ross (2010).)

Compare Empirical CDF (ECDF) to Theoretical CDF

- Given n random variables $X_1, X_2, \ldots, X_n$, (**the data**)
  - each with **CDF** (Cumulative Distribution Function) $F(x) = P(X_k \leq x)$,
  - the **empirical CDF (ECDF)** is

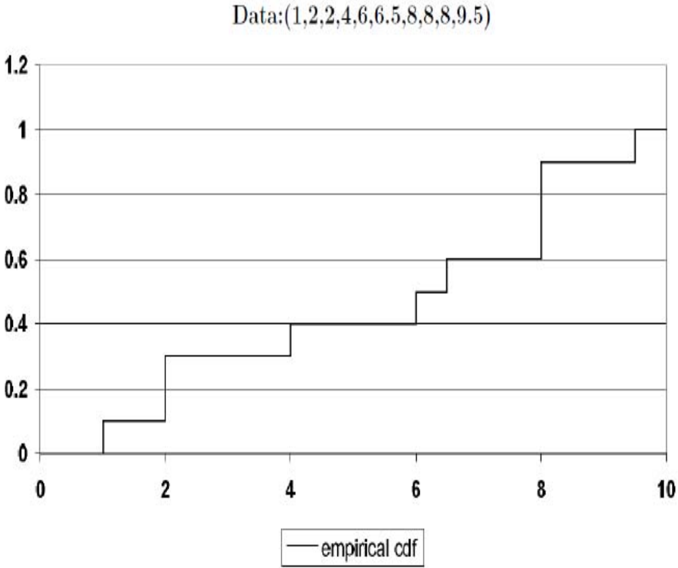
$$\hat{F}_n(x) = \frac{1}{n} \sum_{k=1}^{n} 1_{\{X_k \leq x\}}$$

  ($\hat{F}_n(x)$ is the **proportion** of the n variables less than or equal to x.)

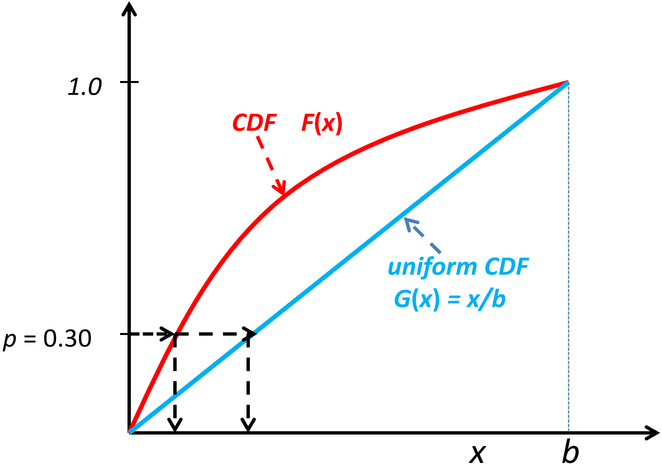
- The ECDF is an estimator of the CDF F.
  - **unbiased**: For each x, $E[\hat{F}_n(x)] = F(x)$.
  - If $\{X_k\}$ are iid, then **consistent**: **absolute difference**

    $D_n \equiv \sup_x |\hat{F}_n(x) - F(x)| \to 0 \quad \text{as} \quad n \to \infty$. (Glivenko-Cantelli Thm.)

Example of an Empirical cdf (ECDF)



Compare Two CDF's



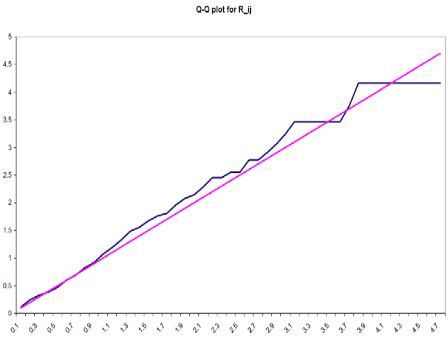
Quantiles $F^{-1}(p)$ and $G^{-1}(p)$

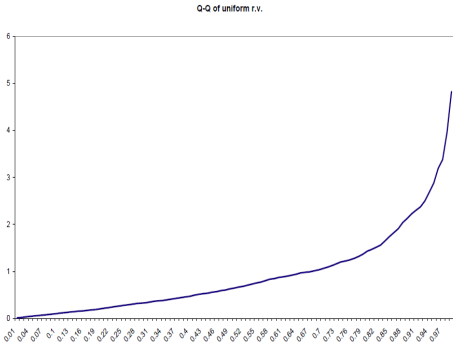
Q-Q Plots: Comparing Two CDF's Via Quantiles

- Given two CDF's F and G,
  1. Consider associated quantile functions (inverses) $F^{-1}(p)$ and $G^{-1}(p)$ for $0 \leq p \leq 1$.
  2. Construct function $h : [0, 1] \to R^2$ mapping p into $(F^{-1}(p), G^{-1}(p))$.
  3. Plot the **image** of this function: $\{(F^{-1}(p), G^{-1}(p)) : 0 \leq p \leq 1\}$.
     - curve in $\mathbb{R} \times \mathbb{R}$ or a function mapping $\mathbb{R}$ into $\mathbb{R}$.
- Common convention for empirical CDF's:
  1. Let $\hat{F}_n^{-1}(/(n+1)) = X_{(k)}$, k^{th} smallest (**order statistic**)
  2. **Q-Q plot** is $\{(\hat{F}_n^{-1}(k/(n+1))), F^{-1}(k/(n+1)) : 1 \leq k \leq n\}$ or $\{(X_{(k)}, F^{-1}(k/(n+1)) : 1 \leq k \leq n\}$.

Example of Good and Bad Q-Q plots

Example. QQ-Plots comparing exponential data (good fit) and uniform data (bad fit) to the exponential distribution.





The Kolmogorov-Smirnov (KS) Statistical Test

- **Null Hypothesis, H0:** We have a sample of size n from a sequence $\{X_k : k \geq 1\}$ of **i.i.d.** random variables with continuous **CDF** F.
- **Alternative Hypothesis, H1:** We have a sample of size n from a another (specified) sequence of random variables.
  - the CDF of X_k might be **not** F.
  - There might be **dependence** among the random variables.
- KS test based on **absolute difference** $D_n \equiv \sup_x |\hat{F}_n(x) - F(x)|$.
  - Observe $D_n = \hat{D}_n$ for the data. Reject if $\hat{D}_n > x_\alpha$, where
  - $P(D_n > x_\alpha | H0) = \alpha = 0.05$ ((**significance level**)
  - Compute **p-value**: $P(D_n > \hat{D}_n | H0)$ (level for rejection)

The KS Test Needs to be Modified for NHPP

- **The KS Test can be applied to test the NHPP.**
  1. Assume that the NHPP has a **piecewise-constant arrival rate function**.
  2. **Exploit the Conditional Uniform (CU) Property** to obtain sequence of i.i.d. random variables uniform on $[0, 1]$ (under H0).
  3. Use code for computing $P(D_n > x_\alpha | H0)$ (e.g. *ksstat* in matlab)
- **Problem**: KS test of NHPP using CU property has **very low power**.
  - **Power:** $P(\text{Reject}\, H0 | H1)$ (**1 -type II error**).
  - Low power means that **alternatives pass too easily!**
- **Solve** by applying KS test after **data transformation**. (Apply KS test after producing new sequence of i.i.d. variables under H0.)

Why does the CU KS Test have low power?

- **The CU transformation focuses on the arrival times instead of the interarrival times.**
  - It is the arrival times that are uniformly distributed on $[0, T]$.
  - Asymptotically, the CU KS test can be shown to have no power. (See §7 and §8 of KW14.)
- **Solve** by applying KS test after **data transformation**. (Apply KS test after producing new sequence of i.i.d. variables under H0.)

The Log KS Test from §3 of Brown et al. (2005)

- Given n ordered arrival times $0 < A_1 < \cdots < A_n < t$ in $[0, t]$, let

$$X_j^{Log} \equiv -(n+1-j) \log_e \left(\frac{t - A_j}{t - A_{j-1}} \right), \quad 1 \le j \le n.$$

- **Under H0:** If these random variables are obtained from a PP over $[0, t]$ using the CU property, then $\{X_j^{Log} : 1 \le j \le n\}$ are n i.i.d. mean-1 exponential random variables. (Proof in §2.2 of KW14a Appendix.)
- The $-\log_e (1 - X_j^{Log})$ are n i.i.d. uniforms on $[0, 1]$.
- The KS test can also be applied in this new setting.
- And the power is greater than direct KS + CU for most alternatives.

The Lewis (1965) KS Test Based on Durbin (1961), Part I

- Given n **ordered arrival times** A_j, $0 < A_1 < \cdots < A_n$, from a Poisson process over $[0, t]$, apply the Conditional Uniform (CU) property to deduce that $U_{(j)} \equiv A_j/t$ are n **ordered uniforms** in $[0, 1]$.
- Construct the **successive intervals between** these ordered uniforms, getting $C_1 = U_{(1)}, C_j = U_{(j)} - U_{(j)}$ and $C_{n+1} = 1 - U_{(n)}$.
- Let $C_{(j)}$ be the associated **ordered intervals** from $\{C_j\}$, so that $0 < C_{(1)} < C_{(2)} < \cdots < C_{(n+1)} < 1$.
- Finally, let $Z_j = (n + 2 - j)(C_{(j)} - C_{(j-1)})$ be the **scaled intervals between**), and let $S_k = Z_1 + \cdots + Z_k$ be the **associated partial sums**.

The Lewis (1965) KS Test Based on Durbin (1961), Part II

- Remarkably, Durbin (1961) showed that **under H0**, $(Z_1, \ldots, Z_n)$ is distributed the same as $(C_1, \ldots, C_n)$.
- Hence, $(S_1, \ldots, S_n)$ is distributed the same as $(U_{(1)}, \ldots, U_{(n)})$.
- Hence, $\hat{F}_n(x) = n^{-1} \sum_{k=1}^{n} 1_{\{S_k \leq x\}}$ is ECDF of uniform CDF, i.e., the ECDF of i.i.d. random variables uniformly distributed on $[0, 1]$.
- Hence we can apply KS test under H0: i.i.d. uniforms on $[0, 1]$.
- Why? **The Lewis KS test has even more power!** Under alternative hypotheses, the constructed ECDF tends to be more distant from the uniform CDF $F(x) = x$.

Sanity Check

$$
\begin{aligned}
S_k &\equiv \sum_{j=1}^{k} Z_j = \sum_{j=1}^{k} (n+2-j)(C_{(j)} - C_{(j-1)}) \\
&= (n+1)C_{(1)} + n(C_{(2)} - C_{(1)}) + (n-1)(C_{(3)} - C_{(2)}) \\
&\qquad + \cdots + (n+2-k)(C_{(k)} - C_{(k-1)}) = C_{(1)} + C_{(2)} + \cdots + C_{(k)} \\
&= U_{(1)} + (U_{(2)} - U_{(1)}) + (U_{(3)} - U_{(2)}) + \cdots + (U_{(k)} - U_{(k-1)}) \\
&= U_{(k)} = C_1 + \cdots + C_k \leq 1, \quad 1 \leq k \leq n+1.
\end{aligned}
$$

Hence, $Z_k \geq 0$ and $\sum_{j=1}^{n+1} Z_j = 1$. But that does not explain the key property that $(Z_1, \ldots, Z_n)$ is distributed the same as $(C_1, \ldots, C_n)$ under the null hypothesis.

Example: Different KS Tests Applied to an Alternative

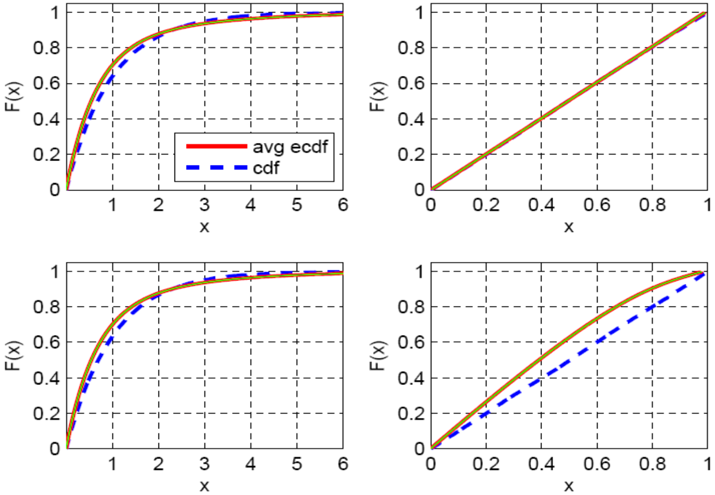
Simulation Experiment: Apply the KS test to the **alternative:** a non-Poisson renewal process with interarrival times having an H_2 (hyperexponential) CDF (mixture of two exponentials) with a squared coefficient of variation $c_X^2 = Var(X)/(E[X])^2 = 2.0$.

Table: **Performance of alternative KS tests** of a rate-1 Poisson process for the time interval $[0, 200]$ with significance level $\alpha = 0.05$: the case of a renewal process with H_2 interarrival times having $c_X^2 = 2$, based on 10^4 replications..

KS test	Lewis	Standard	Log	CU
Power	0.94	0.63	0.51	0.28
Average p value	0.01	0.10	0.13	0.23

Insightful Plots: Average of ECDF over 10^4 Replications

H_2 $(c^2 = 2)$: Standard KS, Conditional-Uniform, Lewis, Log Tests (clockwise)

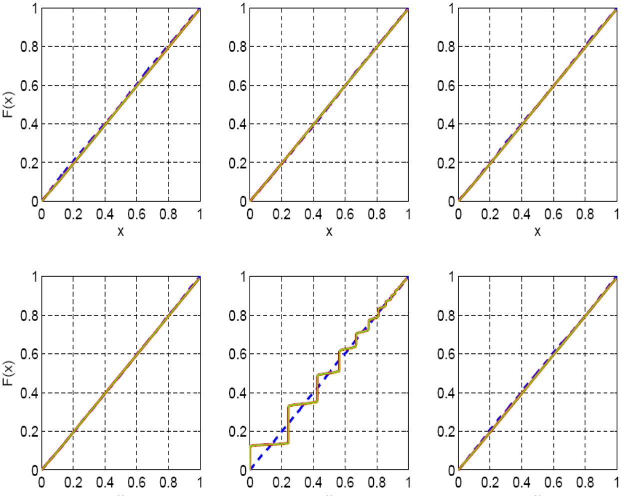


What About Call Center Data

- The banking call center data passes all the KS tests for a NHPP, as described above,
- **provided** we adjust for rounding to nearest second.
- We adjust by **un-rounding**, i.e., by adding small independent uniform random variables, to undo the rounding.
  - Rounding causes rejection by Lewis KS test (but not CU KS test).
  - Unrounding avoids problem.
  - Unrounding does not change non-Poisson into Poisson.

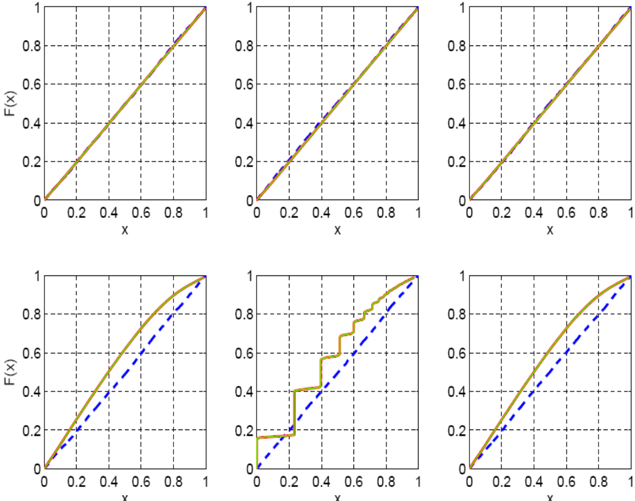
Insightful Plots: Rounding a Poisson Process

Figure 1 Comparison of the average ecdf for a rate-1000 Poisson process. From top to bottom: CU, Lewis test. From left to right: Raw, Rounded, Un-rounded.



Insightful Plots: Rounding an H_2 Renewal Process

Figure 2 Comparison of the average ecdf of a rate-1000 arrival process with H_2 interarrival times. From top to bottom: CU, Lewis test. From left to right: Raw, Rounded, Un-rounded.



References

- L. Brown et al. Statistical Analysis of a Telephone Call Center: A Queueing Science Perspective. *Journal of the American Statistical Association* (JASA) 100 (2005) 36-50.
- J. Durbin. Some Methods for Constructing Exact Tests. *Biometrika* 48 (1961) 41-55.
- S-H. Kim and WW, First NHPP paper. Choosing Arrival Process Models for Service Systems: Tests of a Nonhomogeneous Poisson Process. *Naval Research Logistics* 61 (2014a) 66-90.
- P. A. W. Lewis. Some Results on Tests for Poisson Processes. *Biometrika* 52 (1965) 67-77.

More Work

- S-H. Kim and WW, Second NHPP paper. Are Call Center and Hospital Arrivals Well Modeled by Nonhomogeneous Poisson Processes? *Manufacturing and Service Operations Management* 16 (2014b) No. 3, 464-480.
  - The impact of data rounding and correcting for it.
  - How to choose subintervals to make arrival rate piecewise constant.
  - Testing for over-dispersion in the daily totals. (in the call center data)
- S-H. Kim, Ponni Vel, WW and W. C. Cha Third NHPP paper. Poisson and Non-Poisson Properties in Appointment-Generated Arrival Processes: the Case of an Endocrinology Clinic. *Operations Research Letters* 43 (2015) 247-253. (more next class)