Little's Law*

$$L = \lambda W$$

- L time average number in *L*ine or system
- λ arrival rate
- W average **W**aiting time per customer

*J. D. C. Little, **A proof of the queueing formula: $L = \lambda W$**, *Operations Research* 9 (1961) 383-387.

(See written class lecture notes for more references.)

Outline

- Two Examples
  - Unassigned problems from Homework 2
- Careful Statement
- Sketch of Proof (Explanation)
- Two More Examples
  - More unassigned problems from Homework 2
- More on Proof Details (written notes)

Little's Law EX 1 (HW2 Q3)

- A mad scientist has been studying the passage of insects through a certain cubic meter of air in central Minnesota, using automated instruments to continuously monitor insect positions.
- Her measurements show that, during the calendar year 1990, insects crossed the boundary of the "invisible cube" at an overall rate of 0.061 per hour, either going in or going out, and that the average number of insects in the cube was 0.0082.

- **Q:** What was the average duration of an insect visit to the cube during 1990?

Little's Law EX 1 (HW2 Q3)

- Since the scientist counts both "going in" and "going out", the average arrival rate is

λ = (0.061/2) insects/hour = 0.0305 insects/hour.

The average number of insects in the cube is **L** = 0.0082. Therefore, the average duration of an insect visit is

$W = L / \lambda$

= (0.0082 insects) / (0.0306 insects/hour) = 0.269 hours.

Little’s Law EX 2 (HW2 Q6)

- It is known that 100 candidates on average pass the annual qualification exam for accountants in Israel. An accountant works for 20 years on average (until retirement or professional change).

- **Q1:** Approximately how many accountants will be employed in Israel in 2050?
- **Q2:** Briefly explain the assumptions that are used in your solution.

Little's Law EX 2 (HW2 Q6)

- In 2050, approximately

$$L = \lambda W = 100 \times 20 = 2000$$

accountants will be employed.

- Assumptions:
  - λ and L will be constant till 2050.
  - The edge effects cancel.

Careful Statement

Basic Definitions

- A_k arrival time of customer k (if there after time 0)
- D_k departure time of customer k
- W_k waiting time of customer k
  - $W_k = D_k - A_k$
- $R(0)$ number of remaining customers at time 0
  - arrived before time 0
- $A(t)$ number of new arrivals in $[0,t]$
  - $A(t) = \max\{k\text{: } A_k \leq t\} - R(0)$
- $L(t)$ number in system at time t;
  - So $L(0) = R(0) + A(0)$, common case is $A(0) = 0$

Averages over [*0,t*]

Time Averages:

$$\bar{\lambda}(t) \equiv t^{-1} A(t), \quad \bar{L}(t) \equiv t^{-1} \int_0^t L(s)\, ds,$$

Customer Average:

$$\bar{W}(t) \equiv (1/A(t)) \sum_{k=R(0)+1}^{R(0)+A(t)} W_k,$$

(among new arrivals in [*0,t*])

Averages Among First *n* Arrivals

$T_n = A_{n + R(0)}$ arrival epoch of n^{th} new arrival

Time Averages (over $[0,T_n]$):

$$\bar{\lambda}_n \equiv n/T_n, \quad \bar{L}_n \equiv (1/T_n) \int_0^{T_n} L(s)\, ds,$$

Customer Average:

$$\bar{W}_n \equiv n^{-1} \sum_{k=R(0)+1}^{R(0)+n} W_k$$

Theorem (Little's law for limits of averages)

If

$$\bar{\lambda}(t) \quad \rightarrow \quad \lambda \quad as \quad t \rightarrow \infty \quad and \quad \bar{W}_n \rightarrow W \quad as \quad n \rightarrow \infty,$$

where $0 < \lambda < \infty \quad and \quad W < \infty,$

then

$$(\bar{L}(t), \bar{\lambda}(t), \bar{W}(t)) \quad \rightarrow \quad (L, \lambda, W) \quad as \quad t \rightarrow \infty \quad and$$
$$(\bar{L}_n, \bar{\lambda}_n, \bar{W}_n) \quad \rightarrow \quad (L, \lambda, W) \quad as \quad n \rightarrow \infty,$$

where $L = \lambda W$

(full proof in the written lecture notes.)

Theorem (Little's law for Steady State)

If, for a ***stochastic model****, the arrival rate* λ *is well defined, and limiting distributions exist, i.e.,*

$$L(t) \to L(\infty) \text{ as } t \to \infty$$

and

$$W_n \to W_\infty \text{ as } n \to \infty,$$

(*where* $\to$ means convergence in distribution)

then $E[L(\infty)] = \lambda\, E[W_\infty]$

Sketch of Proof

for the Relation among limits for averages

Proof Idea

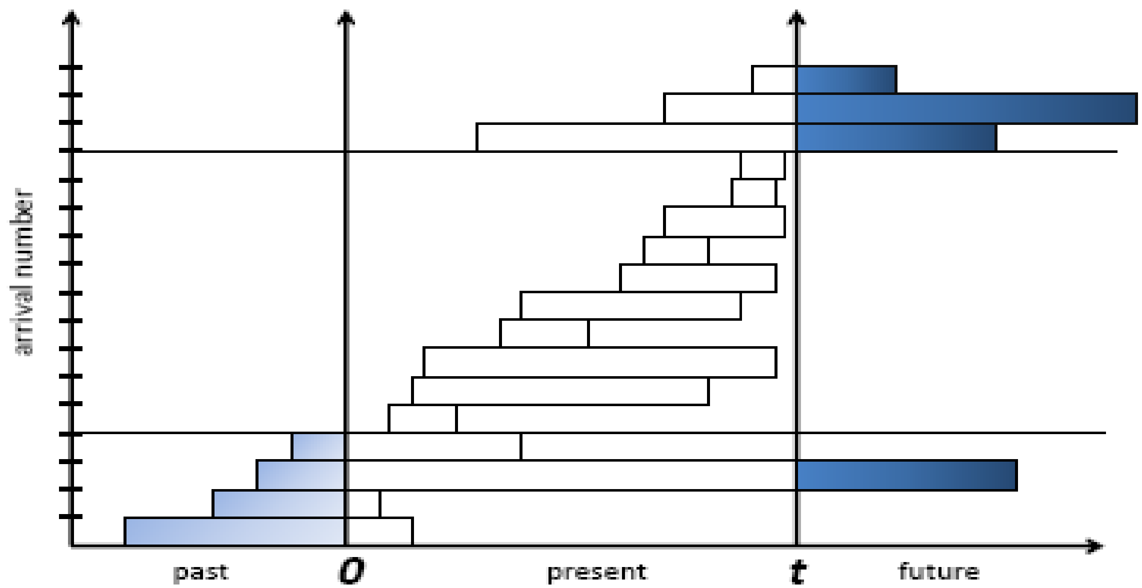


Figure 1: The total work in the system during the interval $[0, t]$ with edge effects: including arrivals before time 0 and departures after time t.

Proof Idea Continued

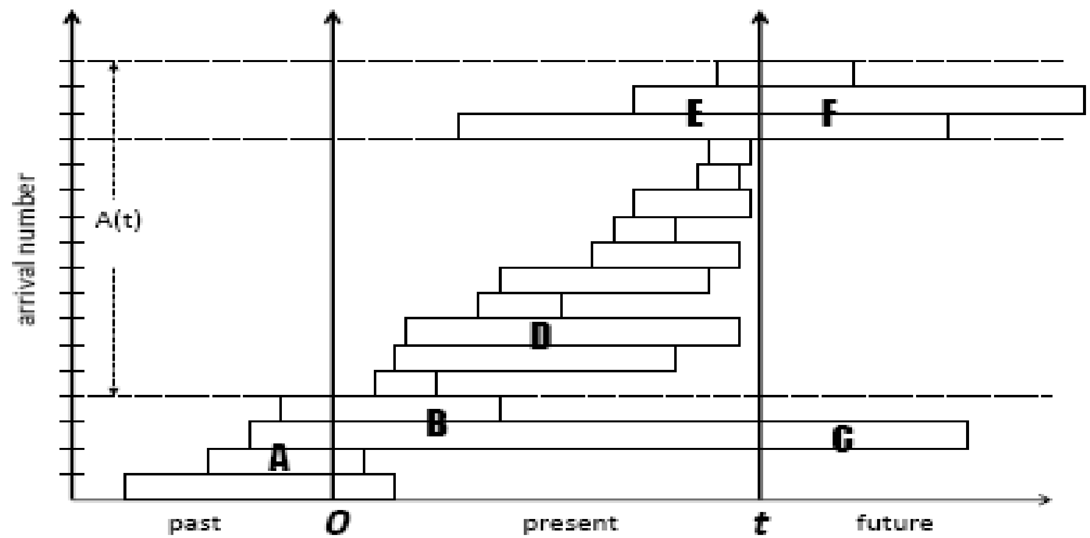


Figure 2: Six regions: waiting times (i) of customers that both arrive and depart inside $[0, t]$ (D), (ii) of arrivals before time 0 ($A \cup B \cup C$) and (iii) of departures after time t ($C \cup E \cup F$).

The main part is region *D.*

(The only part if System Starts and Ends Empty)

Compute Area of Region *D* in Two Ways

Area of D = $\int_0^t L(s)\,ds$ = $\sum_{k=R(0)+1}^{R(0)+A(t)} W_k$

So: $\bar{L}(t) \equiv t^{-1} \int_0^t L(s)\,ds$

= $\bar{\lambda}(t) \equiv t^{-1} A(t)$ x $\bar{W}(t) \equiv (1/A(t)) \sum_{k=R(0)+1}^{R(0)+A(t)} W_k$

(Equality if System Starts and Ends Empty)

How to Apply with Measurements?

$$\bar{L}(t) \quad \approx \quad \bar{\lambda}(t)\bar{W}(t)$$

and

$$\bar{L}_n \quad \approx \quad \bar{\lambda}_n \bar{W}_n$$

(These are **approximations**; equality if begin and end empty. More generally, we may want to evaluate with statistical analysis, e.g., by estimating confidence intervals; see Kim and WW (2012a,b); next class.)

TWO MORE EXAMPLES

Unassigned ones in Homework 2

Others in Recitation 2 and Homework 2

Little's Law EX 3 (HW2 Q7)

- Assume that ***K* judges** work at a court. The following data was collected for every judge:
  - $L_{i,j}$: Number of pending cases that await decision of judge j, $1 \le j \le K$, at the end of the month i, $1 \le i \le 12$.
  - $\lambda_{i,j}$: Number of cases that judge j, $1 \le j \le K$, resolved during month i, $1 \le i \le 12$.

- The head-judge would like to estimate the average sojourn time of a case in the court, <u>per judge and overall</u>.

- **Q:** How can he use the above data without additional measurements? Explain your method and outline restrictions.

Little's Law EX 3 (HW2 Q7)

- **The average sojourn time for judge *j***
  - We can estimate the average number of pending cases by

$$L_j \quad = \quad \sum_{i=1,\ldots,12} L_{i,j} \;/\; 12,$$

  - And the average number of cases that the judge resolves per month by

$$\lambda_j \quad = \quad \sum_{i=1,\ldots,12} \lambda_{i,j} \;/\; 12,$$

  - Then the average sojourn time of a case at judge j is (in months)

$$W_j \quad = \quad L_j / \lambda_j \quad = \quad \sum_{i=1,\ldots,12} L_{i,j} \;/\; \sum_{i=1,\ldots,12} \lambda_{i,j} \,.$$

Little's Law EX 3 (HW2 Q7)

- **The overall average sojourn time:**

$$\mathbf{W} \quad = \quad \sum_j \sum_{i=1,\ldots,12} \mathbf{L}_{\mathbf{i,j}} \,/\, \sum_j \sum_{i=1,\ldots,12} \boldsymbol{\lambda}_{\mathbf{i,j}} \;.$$

Alternatively,

$$\mathbf{W} \quad = \quad \mathbf{L} \,/\, \boldsymbol{\lambda} \;\mathbf{=}\; \sum_j \mathbf{L}_{\mathbf{j}} \,/\, \sum_j \boldsymbol{\lambda}_{\mathbf{j}}$$

$$= \quad \sum_j \boldsymbol{\lambda}_{\mathbf{j}} \mathbf{W}_{j} \,/\, \boldsymbol{\lambda}$$

Little’s Law EX 4 (HW2 Q8)

- A hospital emergency room (ER) is organized so that all patients register through an initial check-in process. At his/her turn, each patient is seen by a doctor and then exits the process, either with a prescription or with admission to the hospital.

- Currently, 50 people per hour arrive at the ER, 10% of whom are admitted to the hospital. On average, 30 people are waiting to be registered and 40 are registered and waiting to see a doctor. The registration process takes, on average, 2 minutes per patient. Among patients who receive prescriptions, average time spent with a doctor is 5 minutes. Among those admitted to the hospital, average time is 30 minutes.

- **Q1:** On average, how long does a patient stay in the ER?
- **Q2:** On average, how many patients are being examined by doctors?
- **Q3:** On average, how many patients are in the ER?

Little's Law EX 4 (HW2 Q8)

- The ER can be divided into 4 subsystems:
  1. Queue for registration.
  2. Registration.
  3. Queue for doctors.
  4. Doctors.

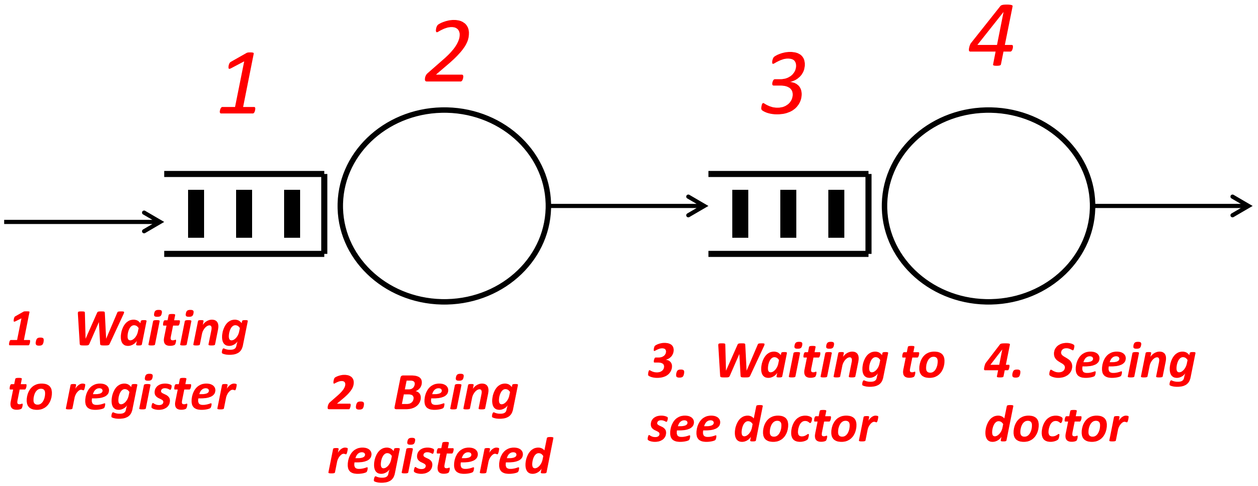
  - The arrival rate λ to all subsystems is 50 per hour = 5/6 per min.

- Denote by W_i and L_i the waiting time and average number of customers in subsystem i. Then,

$$L_1 = 30; \quad W_2 = 2 \text{ min}; \quad L_3 = 40;$$

$$W_4 = (5 \times 0.9) + (30 \times 0.1) = 7.5 \text{ min.}$$

4 stages



$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 =$ 50/hour = 5/6 per minute

L_1 = 30

W_2 = 2 minutes

L_3 = 40

W_4 =(5 x 0.9) + (30 x 0.1) = 7.5 minutes

So

$W_1 = L_1/\lambda_1 =$ 30/(5/6) = 36 minutes

$L_2 = \lambda_2 W_2 =$ (5/6) x 2 = 5/3 patients

$W_3 = L_3/\lambda_3 =$ 40/(5/6) = 48 minutes

$L_4 = \lambda_4 W_4 =$ (5/6) x (15/2) = 75/12 = 6.25 patients (answers Q2)

Little's Law EX 4 (HW2 Q8)

- From above, we can answer the three questions:

- If we denote by **L** and **W** the average number of customers and the average waiting time in ER respectively, then

  L = $L_1 + L_2 + L_3 + L_4$ = 77.9 (answers Q3)

  W = $W_1 + W_2 + W_3 + W_4$ =93.5 minutes (answers Q1)

In summary,

- A patient stays 93.5 minutes in the ER on average.
- On average, 6.25 patients are being examined by doctors.
- There are 77.9 patients in the ER on average.