Statistical Analysis with *Little's Law*

IEOR 4615, Lecture 4

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Research paper available in CourseWorks and online

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Based on Lecture by Song-Hee Kim

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Problem 1: Estimate Expected Waiting Time $W \equiv E[W_\infty]$

From direct measurements:

- **Observe** waiting times $W_{i,j}$ for customer j, $1 \leq j \leq n$, during same given time interval on day i, $1 \leq i \leq m$.
- Average waiting at this time on day i is $\bar{W}_n^{(i)} \equiv \frac{1}{n} \sum_{j=1}^{n} W_{i,j}$.
- Average waiting over all days is $\bar{W}_{n,m} \equiv \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} W_{i,j} = \frac{1}{m} \sum_{j=1}^{m} \bar{W}_n^{(i)}$
- How to estimate **confidence interval (CI)** for $W \equiv E[W_\infty]$? For example, $[\bar{W} - \bar{h}, \bar{W} + \bar{h}]$, where $\bar{h}$ is the CI halfwidth.
- With all data? On any one day?

Problem 2: Apply $L = \lambda W$ to Estimate $W \equiv E[W_\infty]$

- **Observe** $L(s)$ over $0 \leq s \leq t$, but not waiting times.
- Given $\{L(s)\}$, we can directly observe the arrivals and departures.
- We can can easily estimate λ and L.
- But we typically cannot determine W_k, because the items need not depart in the same order they arrived.
- Nevertheless, we can estimate W by $W = L/\lambda$ using our estimates.
- How to estimate **confidence interval (CI)**? How to eliminate **bias**?

$L = \lambda W$

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$L = \lambda W$: Theory

$$L = \lambda W$$

(i) mean values of stationary distributions

$$\mathbb{E}L(\infty) = \lambda \mathbb{E}W_\infty$$

(ii) relation among limits of averages (limiting sample path averages)

(avg number in system) = (arrival rate) (avg time spent)

$$\left(\lim_{t\to\infty} t^{-1}\int_0^t L(s)\,ds\right) = \left(\lim_{t\to\infty} t^{-1}A(t)\right)\left(\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} W_k\right)$$

$L = \lambda W$: **Measurements**

$$L = \lambda W$$

(avg number in system) $\approx$ (arrival rate) (avg time spent)

$$\bar{L}(t) \approx \bar{\lambda}(t) \quad \bar{W}(t)$$

$$\frac{\int_0^t L(s)\, ds}{t} \approx \frac{A(t)}{t} \quad \frac{\sum_{k=R(0)+1}^{R(0)+A(t)} W_k}{A(t)}$$

NOTE: finite averages over $[0, t]$; $=$ if start and end empty

Main Idea

- Apply Statistics with Finite Averages from Data

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- Estimate and remove bias

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- Apply Statistics with Finite Averages from Data
- Estimate and remove bias
- Estimate confidence intervals
  i. Stationary Framework - Method of batch means
  ii. Nonstationary Framework - Sample averages over multiple days

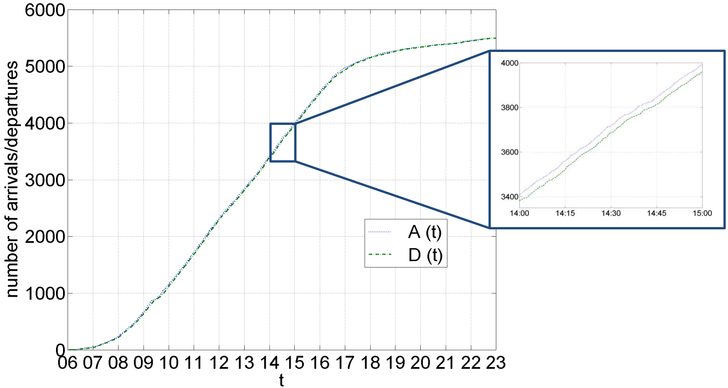
Data

A US bank call center data from (Mandelbaum 2012)*

- about $60,000$ calls (of all types) handled by agents on weekdays
- one type of customers (*Summit*)
- 17-hour period from 6 am to 11 pm, referred to as $[6, 23]$
- Friday, May 25, 2001: 5749 call arrivals of which 253 abandoned before starting service
- 18 weekdays similar to May 25, 2001

* We thank Professor Avi Mandelbaum and the SEELab at the Technion.

Using $L = \lambda W$: One Day in a Banking Call Center



- $\lambda(t)$ is not stationary over the entire day

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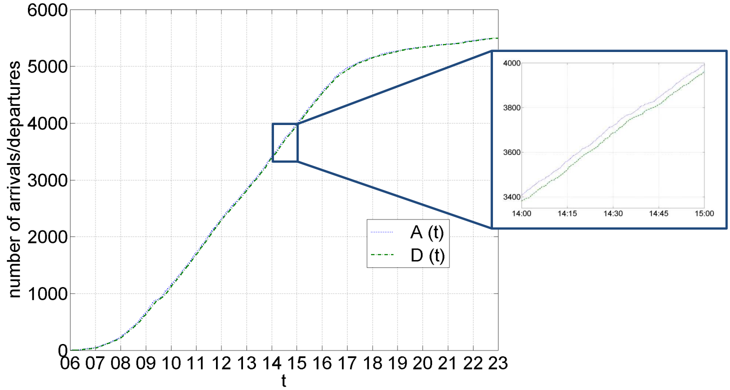

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- $\lambda(t)$ is approximately stationary over the middle part, $[10, 16]$

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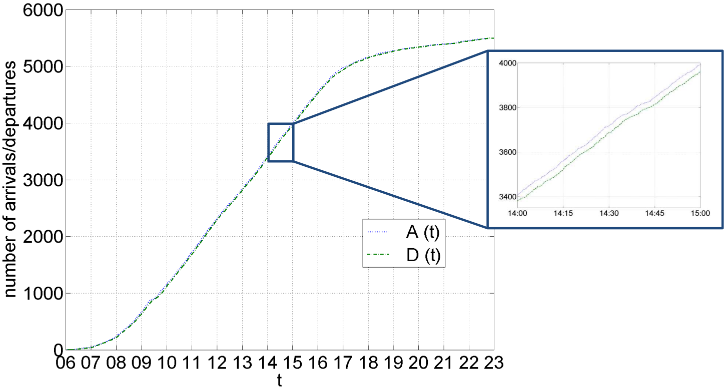
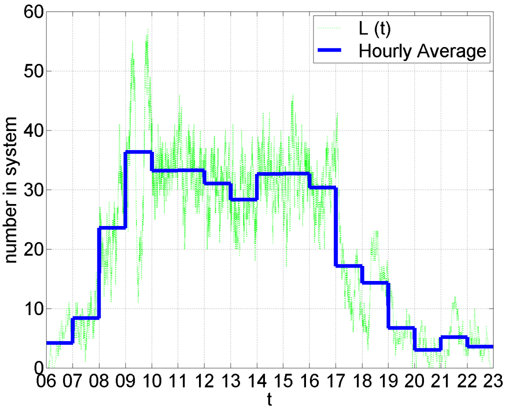


- $\lambda(t)$ is not stationary over the entire day
- $\lambda(t)$ is approximately stationary over the middle part, $[10, 16]$
  - Stationarity confirmed by turning points test, difference-sign test and rank test for randomness [Brockwell and Davis, 1991].

Using $L = \lambda W$: One Day in a Banking Call Center (2)



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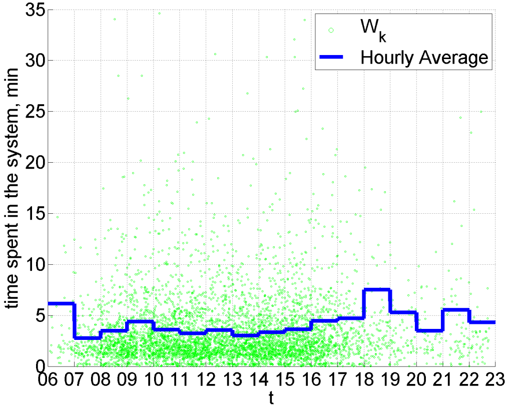
Using $L = \lambda W$: One Day in a Banking Call Center (3)



- W_k is approximately stationary over the entire day

Canonical Problem

The essence of a typical application:

- observe $L(s)$ over $0 \leq s \leq t$, but not waiting times
- given $\{L(s)\}$, can directly observe the arrivals and departures
- can easily estimate λ and L
- typically cannot determine W_k, because the items need not depart in the same order they arrived
- nevertheless, can estimate W by $W = L/\lambda$ using our estimates

Canonical Problem: Direct and Indirect Estimators

Direct Estimators

$$\bar{\lambda}(t) \equiv \frac{A(t)}{t}, \quad \bar{L}(t) \equiv \frac{\int_0^t L(s)\,ds}{t} \quad \text{and} \quad \bar{W}(t) \equiv \frac{\sum_{k=R(0)+1}^{R(0)+A(t)} W_k}{A(t)}.$$

Indirect Estimators

$$\bar{\lambda}_{L,W}(t) \equiv \frac{\bar{L}(t)}{\bar{W}(t)}, \quad \bar{L}_{\lambda,W}(t) \equiv \bar{\lambda}(t)\bar{W}(t) \quad \text{and} \quad \bar{W}_{\lambda,L}(t) \equiv \frac{\bar{L}(t)}{\bar{\lambda}(t)}.$$

We want to use $\bar{W}_{\lambda,L}(t)$ as a substitute for $\bar{W}(t)$.

Estimating W given L and λ

Over the entire day $[6, 23]$:

$$\bar{L}_{[6,23]} = 20.2 \pm 6.1, \quad \bar{\lambda}_{[6,23]} = 5.39 \pm 1.84 \quad \longrightarrow \quad \bar{W}_{[6,23];L,\lambda} = 3.75$$

$\longrightarrow$ Averages do not have much meaning.

$\longrightarrow$ Halfwidths reveal nonstationarity.

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$\longrightarrow$ Halfwidths reveal nonstationarity.

Over an approximately stationary interval $[10, 16]$:

$$\bar{L}_{[10,16]} = 31.9 \pm 1.9, \quad \bar{\lambda}_{[10,16]} = 9.44 \pm 0.49 \quad \longrightarrow \quad \bar{W}_{[10,16];L,\lambda} = 3.38$$

$\longrightarrow$ $\bar{L}_{[10,16]}$ and $\bar{\lambda}_{[10,16]}$ are very different from $\bar{L}_{[6,23]}$ and $\bar{\lambda}_{[6,23]}$.

$\longrightarrow$ System is not empty at 10 am and 4 pm.

$\longrightarrow$ Two errors cancel for W ($\bar{W}_{[10,16]} = 3.38$).

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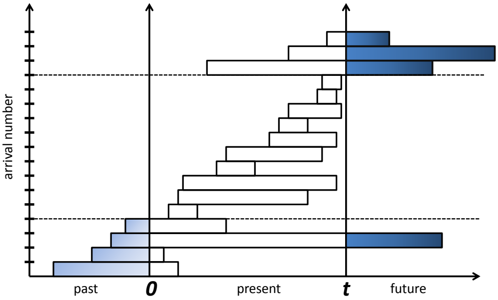
3. For a stationary interval, how well do we know L, λ and W by $\bar{L}(t)$, $\bar{\lambda}(t)$ and $\bar{W}(t)$?

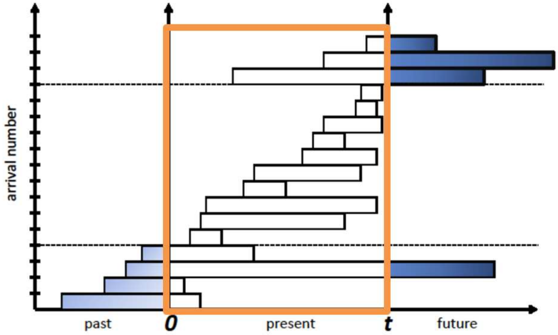
4. What can we do in a nonstationary setting?

Total Work in the System



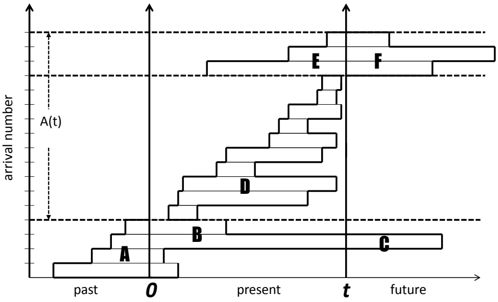
- a bar of height 1 for each customer k (width $= W_k$)
- for $0 \leq s \leq t$, the number of bars above any time s is $L(s)$
- order: arrived before 0; arrive after 0 and depart before t and arrive after 0 but depart after t

Alternative Definitions to Force Equality: The Inside View



- $A_i(t) \equiv R(0) + A(t), \quad t \geq 0$ - made bigger
- $W_n^i \equiv (D_n \wedge t) - (A_n \vee 0), \quad n \geq 1$ - made shorter
- $\bar{L}(t) = \bar{\lambda}_i(t)\bar{W}_i(t)$ (Little 2011, Buzen 1976, Denning and Buzen 1978)
- distorts meaning

Two Cumulative Processes



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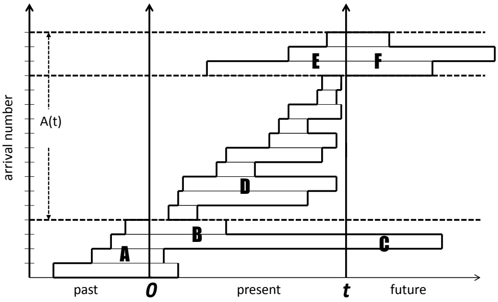
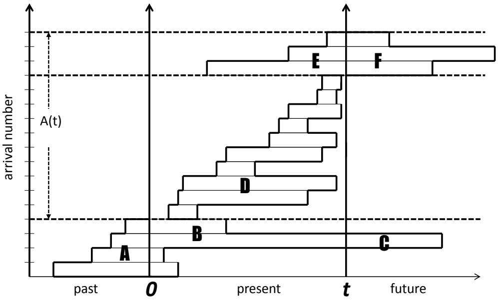


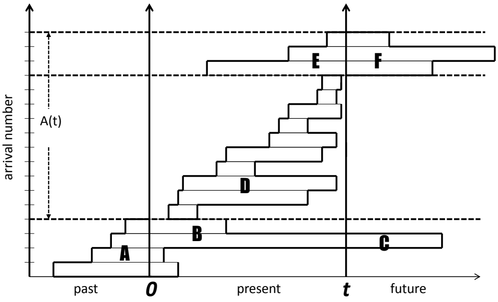
- $C_L(t) = |B \cup D \cup E| \equiv \int_0^t L(s)\, ds = \bar{L}(t)t$

Two Cumulative Processes



- $C_L(t) = |B \cup D \cup E| \equiv \int_0^t L(s)\,ds = \bar{L}(t)t$
- $C_W(t) = |D \cup E \cup F| \equiv \sum_{k=R(0)+1}^{R(0)+A(t)} W_k = \bar{W}(t)A(t)$

Two Cumulative Processes



- $C_L(t) = |B \cup D \cup E| \equiv \int_0^t L(s)\, ds = \bar{L}(t)t$
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- $C_L(t) - C_W(t) = |B \cup D \cup E| - |D \cup E \cup F| = |B| - |F|$

Finite-time version of Little's Law (Jewell 1967)

Theorem

If $R(0) = L(t) = 0$, *then* $\bar{L}(t) = \bar{\lambda}(t)\bar{W}(t)$.

Proof: In general,

$$\bar{L}(t) \equiv \frac{\int_0^t L(s)\,ds}{t} = \frac{C_L(t)}{t}$$

$$\bar{\lambda}(t)\bar{W}(t) \equiv \left(\frac{A(t)}{t}\right)\left(\frac{\sum_{k=R(0)+1}^{R(0)+A(t)} W_k}{A(t)}\right) = \left(\frac{A(t)}{t}\right)\left(\frac{C_W(t)}{A(t)}\right)$$

Under the condition, $C_L(t) = C_W(t)$, so that

$$\bar{L}(t) \equiv \frac{C_L(t)}{t} = \frac{C_W(t)}{t} = \bar{\lambda}(t)\bar{W}(t).$$

Extended finite-time version of Little's Law

Theorem

The empirical averages are related by

$$
\begin{aligned}
\Delta_L(t) &\equiv \bar{L}_{\lambda,W}(t) - \bar{L}(t) = \frac{|F| - |B|}{t}, \\
\Delta_W(t) &\equiv \bar{W}_{L,\lambda}(t) - \bar{W}(t) = \frac{|B| - |F|}{A(t)} = -\frac{\Delta_L(t)}{\bar{\lambda}(t)}, \\
\Delta_\lambda(t) &\equiv \bar{\lambda}_{L,W}(t) - \bar{\lambda}(t) = \left(\frac{|B| - |F|}{|D| + |E| + |F|}\right)\bar{\lambda}(t) = -\frac{\Delta_L(t)}{\bar{W}(t)},
\end{aligned}
$$

where $|B|$ is the area of the region B.

Estimating and Reducing the BIAS

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- An unbiased estimator based on the observed data over $[0, t]$ $(\mathscr{O}_t)$ is $\bar{W}_{L,\lambda,u}(t) \equiv \bar{W}_{L,\lambda}(t) - E[\Delta_W(t)|\mathscr{O}_t]$.

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- Since $\Delta_W(t) \equiv \bar{W}_{L,\lambda}(t) - \bar{W}(t) = \frac{|B|-|F|}{A(t)}$, where
  - $|B|$: the total remaining work at time 0
  - $|F|$: the total remaining work at time t

  a natural Approximation is

$$E[\Delta_W(t)|\mathscr{O}_t] \approx \frac{(R(0) - L(t))\bar{W}_{L,\lambda}(t)}{A(t)}$$

Estimating and Reducing the BIAS

- An unbiased estimator based on the observed data over $[0,t]$ ($\mathscr{O}_t$) is $\bar{W}_{L,\lambda,u}(t) \equiv \bar{W}_{L,\lambda}(t) - E[\Delta_W(t)|\mathscr{O}_t]$.

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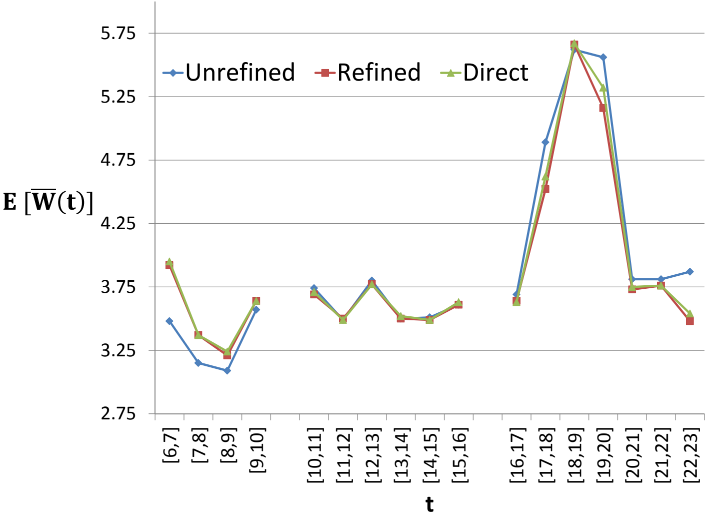
  a natural Approximation is

$$E[\Delta_W(t)|\mathscr{O}_t] \approx \frac{(R(0)-L(t))\bar{W}_{L,\lambda}(t)}{A(t)}$$

$$\rightarrow \bar{W}_{L,\lambda,r}(t) \equiv \bar{W}_{L,\lambda}(t)\left(1 - \frac{R(0)-L(t)}{A(t)}\right).$$

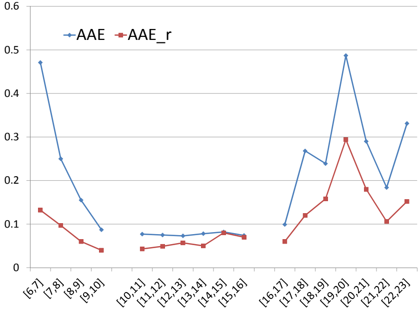
Estimating and Reducing the BIAS: Call Center EX

18 weekdays in May:

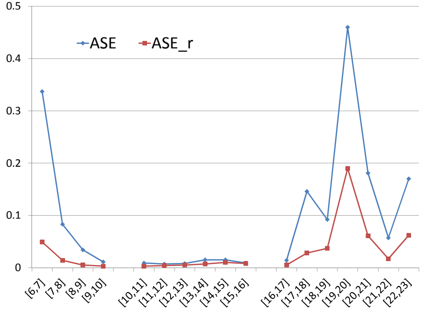


Estimating and Reducing the BIAS - Call Center EX

Avg Absolute Errors (AAE)



Avg Squared Errors (ASE)



→ More bias/bias reduction at the ends of the day when the system is nonstationary.

Constructing CI's: Stationary Framework

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- We take the view that the Little's Law theory applies in a stationary interval and regard the finite averages as estimates of the theoretical values $L \equiv E[L(\infty)]$, λ and $W \equiv E[W_\infty]$. (We uses the relations among the steady-state mean values.)

Constructing CI's: a Central Limit Theorem (CLT)

Constructing CI's: a Central Limit Theorem (CLT)

(X_L, X_λ, X_W) is essentially a two-dimensional mean-zero multivariate Gaussian random vector.

Theorem (A CLT Version of $L = \lambda W$ (Glynn and Whitt 1986))

Direct estimators and indirect estimators converge in distribution jointly and the indirect estimators assume the same values in the limit as the direct estimators. That is,

$$(\hat{L}(t), \hat{\lambda}(t), \hat{W}(t), \hat{L}_{\lambda,W}(t), \hat{\lambda}_{L,W}(t), \hat{W}_{L,\lambda}(t)) \Rightarrow (X_L, X_\lambda, X_W, X_L, X_\lambda, X_W) \quad \text{in} \quad \mathbb{R}^6$$

as $t \to \infty$ under very general regularity conditions, where

$$(\hat{L}(t), \hat{\lambda}(t), \hat{W}(t)) \equiv \sqrt{t}\left(\bar{L}(t) - L, \bar{\lambda}(t) - \lambda, \bar{W}(t) - W\right),$$
$$(\hat{L}_{\lambda,W}(t), \hat{\lambda}_{L,W}(t), \hat{W}_{L,\lambda}(t)) \equiv \sqrt{t}\left(\bar{L}_{\lambda,W}(t) - L, \bar{\lambda}_{L,W}(t) - \lambda, \bar{W}_{L,\lambda}(t) - W\right).$$

Constructing CI's: The Method of Batch Means

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1. Use sample path segment $\{(A(s), L(s)) : 0 \leq s \leq t\}$ over $[0, t]$
2. Divide $[0, t]$ into m intervals $[\frac{(k-1)t}{m}, \frac{kt}{m}]$, $1 \leq k \leq m$
3. Compute batch averages, $\bar{A}_k(t, m)$, $\bar{L}_k(t, m)$ and $\bar{W}_{L,\lambda,k}(t, m) \equiv \frac{\bar{L}_k(t,m)}{\bar{\lambda}_k(t,m)}$
4. $\bar{W}^{(m)}_{L,\lambda}(t) \equiv \frac{1}{m}\sum_{k=1}^{m} \bar{W}_{L,\lambda,k}(t, m)$, $S^2_{(m)}(t) \equiv \frac{1}{m-1}\sum_{k=1}^{m}(\bar{W}_{L,\lambda,k}(t, m) - \bar{W}^{(m)}_{L,\lambda}(t))^2$
5. Construct a two-sided 95% CI based on the Student-t dist.
   - $\bar{W}^{(m)}_{L,\lambda}(t) \pm t_{0.025,m-1}\sqrt{\frac{S^2_{(m)}(t)}{m}}$

Constructing CI's: What We Saw Before

Over the entire day [6, 23]:

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→ Averages do not have much meaning.

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Over an approximately stationary interval [10, 16]:

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→ $\bar{L}_{[10,16]}$ and $\bar{\lambda}_{[10,16]}$ are very different from $\bar{L}_{[6,23]}$ and $\bar{\lambda}_{[6,23]}$.

→ System is not empty at 10 am and 4 pm.

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Constructing CI's: Function of Batch Size in Call Center

Direct versus Indirect Estimates for Each Hour of the Day

Interval	m	$\bar{L}^{(m)}(t)$	$\bar{\lambda}^{(m)}(t)$	$\bar{W}^{(m)}(t)$	$\bar{W}_{L,\lambda}^{(m)}(t)$
[10,16]	5	31.9 ± 1.9	9.44 ± 0.49	3.38 ± 0.22	3.38 ± 0.19
	10	31.9 ± 1.3	9.44 ± 0.36	3.39 ± 0.15	3.38 ± 0.16
	20	31.9 ± 1.0	9.44 ± 0.30	3.39 ± 0.15	3.38 ± 0.11
[14,15]	5	32.6 ± 1.9	9.82 ± 0.82	3.33 ± 0.21	3.33 ± 0.10
	10	32.6 ± 1.6	9.82 ± 0.79	3.33 ± 0.21	3.34 ± 0.16
	20	32.6 ± 1.3	9.82 ± 0.81	3.32 ± 0.23	3.43 ± 0.31

Constructing CI's: Function of Batch Size in Call Center

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	10	32.6 ± 1.6	9.82 ± 0.79	3.33 ± 0.21	3.34 ± 0.16
	20	32.6 ± 1.3	9.82 ± 0.81	3.32 ± 0.23	3.43 ± 0.31

- The approach evidently works.

Constructing CI's: Function of Batch Size in Call Center

Direct versus Indirect Estimates for Each Hour of the Day

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- The approach evidently works.
- How to choose m?

Constructing CI's: Supporting Simulation Experiments

Apply simulation to evaluate the estimation procedure performance for an idealized queueing model of the system.

Constructing CI's: Supporting Simulation Experiments

Apply simulation to evaluate the estimation procedure performance for an idealized queueing model of the system.

- Estimate CI coverage based on 1000 replications.
- CIs close to 95%. Be conservative and choose $m = 5$.

case	m	$\bar{L}^{(m)}(t)$	$\bar{\lambda}^{(m)}(t)$	$\bar{W}^{(m)}(t)$	cov.	$\bar{W}^{(m)}_{L,\lambda}(t)$	cov.
$\beta = \infty$	5	31.5 ± 2.0	9.33 ± 0.42	3.38 ± 0.15	95.1%	3.38 ± 0.15	95.4%
$(M_t/M/\infty)$	10	31.5 ± 1.6	9.33 ± 0.35	3.38 ± 0.13	95.0%	3.38 ± 0.13	95.7%
	20	31.5 ± 1.4	9.33 ± 0.33	3.38 ± 0.12	94.4%	3.38 ± 0.12	95.3%
$\beta = 1.0$	5	32.1 ± 2.6	9.33 ± 0.42	3.44 ± 0.21	95.0%	3.44 ± 0.21	95.3%
$(M_t/M/s_t)$	10	32.1 ± 2.1	9.33 ± 0.35	3.44 ± 0.17	93.2%	3.44 ± 0.17	93.5%
	20	32.1 ± 1.8	9.33 ± 0.33	3.44 ± 0.15	91.4%	3.44 ± 0.15	92.5%
data [10, 16]	5	31.9 ± 1.9	9.44 ± 0.49	3.38 ± 0.22		3.38 ± 0.19	
(call center)	10	31.9 ± 1.3	9.44 ± 0.36	3.39 ± 0.15		3.38 ± 0.16	
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Constructing CI's: Nonstationary Framework

Whether stationary or not, we can estimate CI's for $E[\bar{W}(t)]$ using sample averages over multiple days, regarding those days as approximately i.i.d.

Constructing CI's: Nonstationary Framework

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Intervals	direct estimator $\bar{W}(t)$	unrefined estimator $\bar{W}_{L,\lambda}(t)$	refined estimator $\bar{W}_{L,\lambda,r}(t)$
$[6, 10]$	3.47 ± 0.22	3.35 ± 0.23	3.47 ± 0.23
$[10, 16]$	3.60 ± 0.11	3.61 ± 0.11	3.60 ± 0.11
$[16, 23]$	4.24 ± 0.26	4.35 ± 0.26	4.22 ± 0.25

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Constructing CI's: Nonstationary Framework

Whether stationary or not, we can estimate CI's for $E[\bar{W}(t)]$ using sample averages over multiple days, regarding those days as approximately i.i.d.

Intervals	direct estimator $\bar{W}(t)$	unrefined estimator $\bar{W}_{L,\lambda}(t)$	refined estimator $\bar{W}_{L,\lambda,r}(t)$
$[6,10]$	3.47 ± 0.22	3.35 ± 0.23	3.47 ± 0.23
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- $\bar{W}_{L,\lambda,r}(t)$ behaves very similar to $\bar{W}(t)$ in all cases.
- $\bar{W}_{L,\lambda}(t)$ performs well in the stationary region $[10,16]$, but shows the impact of bias in nonstationary regions, $[6,10]$ and $[16,23]$.

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6. Next paper: Using time-varying Little's Law: Kim, S., W. Whitt. 2013. Estimating Waiting Times with the Time-Varying Little's Law. *Probability in the Engineering and Informational Sciences* **27** 471–506.

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