1 Gambler's Ruin Problem

Consider a gambler who starts with an initial fortune of \$1 and then on each successive gamble either wins \$1 or loses \$1 independent of the past with probabilities p and $q = 1-p$ respectively. Let X_n denote the total fortune after the n^{th} gamble. The gambler's objective is to reach a total fortune of $\$N$, without first getting *ruined* (running out of money). If the gambler succeeds, then the gambler is said to *win* the game. In any case, the gambler stops playing after winning or getting ruined, whichever happens first. There is nothing special about starting with \$1, more generally the gambler starts with $\$i$ where $0 < i < N$.

$\{X_n\}$ yields a Markov chain (MC) on the state space $\mathcal{S} = \{0, 1, \ldots, N\}$. The transition probabilities are given by $p_{i,i+i} = p,\ p_{i,i-i} = q,\ 0 < i < N$, and both 0 and N are absorbing states, $p_{00} = p_{NN} = 1$.[1]

For example, when $N = 4$ the transition matrix is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

While the game proceeds, this MC forms a simple random walk

$$X_n = \Delta_1 + \cdots + \Delta_n,\ X_0 = i,$$

where $\{\Delta_n\}$ forms an i.i.d. sequence of r.v.s. distributed as $P(\Delta = 1) = p,\ P(\Delta = -1) = q = 1 - p$, and represents the earnings on the succesive gambles.

Since the game stops when either $X_n = 0$ or $X_n = N$, let

$$\tau_i = \min\{n \geq 0 : X_n \in \{0, N\} | X_0 = i\},$$

denote the time at which the game stops when $X_0 = i$. If $X_{\tau_i} = N$, then the gambler wins, if $X_{\tau_i} = 0$, then the gambler is ruined.

Let $P_i = P(X_{\tau_i} = N)$ denote the probability that the gambler wins when $X_0 = i$. Clearly $P_0 = 0$ and $P_N = 1$ by definition, and we next proceed to compute $P_i,\ 1 \leq i \leq N-1$.

The key idea is to condition on the outcome of the first gamble, $\Delta_1 = 1$ or $\Delta_1 = -1$, yielding

$$P_i = pP_{i+1} + qP_{i-1}. \tag{1}$$

The derivation of this recursion is as follows: If $\Delta_1 = 1$, then the gambler's total fortune increases to $X_1 = i+1$ and so by the Markov property the gambler will now win with probability P_{i+1}. Similarly, if $\Delta_1 = -1$, then the gambler's fortune decreases to $X_1 = i - 1$ and so by the Markov property the gambler will now win with probability P_{i-1}. The probabilities corresponding to the two outcomes are p and q yielding (1). Since $p + q = 1$, (1) can be re-written as $pP_i + qP_i = pP_{i+1} + qP_{i-1}$, yielding

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}).$$

[1] There are three communication classes: $C_1 = \{0\},\ C_2 = \{1, \ldots, N-1\},\ C_3 = \{N\}$. C_1 and C_3 are recurrent whereas C_2 is transient.

In particular, $P_2 - P_1 = (q/p)(P_1 - P_0) = (q/p)P_1$ (since $P_0 = 0$), so that $P_3 - P_2 = (q/p)(P_2 - P_1) = (q/p)^2 P_1$, and more generally

$$P_{i+1} - P_i = (\frac{q}{p})^i P_1, \ 0 < i < N.$$

Thus

$$\begin{aligned} P_{i+1} - P_1 &= \sum_{k=1}^{i}(P_{k+1} - P_k) \\ &= \sum_{k=1}^{i}(\frac{q}{p})^k P_1, \end{aligned}$$

yielding

$$\begin{aligned} P_{i+1} &= P_1 + P_1 \sum_{k=1}^{i}(\frac{q}{p})^k = P_1 \sum_{k=0}^{i}(\frac{q}{p})^k \\ &= \begin{cases} P_1 \frac{1-(\frac{q}{p})^{i+1}}{1-(\frac{q}{p})}, & \text{if } p \neq q; \\ P_1(i+1), & \text{if } p = q = 0.5. \end{cases} \end{aligned} \tag{2}$$

(Here we are using the "geometric series" equation $\sum_{n=0}^{i} a^i = \frac{1-a^{i+1}}{1-a}$, for any number a and any integer $i \geq 1$.)

Choosing $i = N - 1$ and using the fact that $P_N = 1$ yields

$$1 = P_N = \begin{cases} P_1 \frac{1-(\frac{q}{p})^N}{1-(\frac{q}{p})}, & \text{if } p \neq q; \\ P_1 N, & \text{if } p = q = 0.5, \end{cases}$$

from which we conclude that

$$P_1 = \begin{cases} \frac{1-\frac{q}{p}}{1-(\frac{q}{p})^N}, & \text{if } p \neq q; \\ \frac{1}{N}, & \text{if } p = q = 0.5, \end{cases}$$

thus obtaining from (2) (after algebra) the solution

$$P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}, & \text{if } p \neq q; \\ \frac{i}{N}, & \text{if } p = q = 0.5. \end{cases} \tag{3}$$

(Note that $1 - P_i$ is the probability of ruin.)

1.1 Becoming infinitely rich or getting ruined

If $p > 0.5$, then $\frac{q}{p} < 1$ and thus from (3)

$$\lim_{N \to \infty} P_i = 1 - (\frac{q}{p})^i > 0, \ p > 0.5. \tag{4}$$

If $p \leq 0.5$, then $\frac{q}{p} \geq 1$ and thus from (3)

$$\lim_{N \to \infty} P_i = 0, \ p \leq 0.5. \tag{5}$$

To interpret the meaning of (4) and (5), suppose that the gambler starting with $X_0 = i$ wishes to continue gambling forever until (if at all) ruined, with the intention of earning as much money as possible. So there is no winning value N; the gambler will only stop if ruined. What will happen?

(4) says that if $p > 0.5$ (each gamble is in his favor), then there is a positive probability that the gambler will never get ruined but instead will become infinitely rich.

(5) says that if $p \leq 0.5$ (each gamble is not in his favor), then with probability one the gambler will get ruined.

Examples

1. John starts with \$2, and $p = 0.6$: What is the probability that John obtains a fortune of $N = 4$ without going broke?

   SOLUTION $i = 2$, $N = 4$ and $q = 1 - p = 0.4$, so $q/p = 2/3$, and we want

   $$P_2 = \frac{1 - (2/3)^2}{1 - (2/3)^4} = 0.91$$

2. What is the probability that John will become infinitely rich?

   SOLUTION

   $1 - (q/p)^i = 1 - (2/3)^2 = 5/9 = 0.56$

3. If John instead started with $i = \$1$, what is the probability that he would go broke?

   SOLUTION

   The probability he becomes infinitely rich is $1-(q/p)^i = 1-(q/p) = 1/3$, so the probability of ruin is $2/3$.

1.2 Applications

Risk insurance business

Consider an insurance company that earns \$1 per day (from interest), but on each day, independent of the past, might suffer a *claim* against it for the amount \$2 with probability $q = 1 - p$. Whenever such a claim is suffered, \$2 is removed from the reserve of money. Thus on the n^{th} day, the net income for that day is exactly Δ_n as in the gamblers' ruin problem: 1 with probability p, -1 with probability q.

If the insurance company starts off initially with a reserve of \$$i \geq 1$, then what is the probability it will eventually get ruined (run out of money)?

The answer is given by (4) and (5): If $p > 0.5$ then the probability is given by $(\frac{q}{p})^i > 0$, whereas if $p \leq 0.5$ ruin will always ocurr. This makes intuitive sense because if $p > 0.5$, then the average net income per day is $E(\Delta) = p - q > 0$, whereas if $p \leq 0.5$, then the average net income per day is $E(\Delta) = p - q \leq 0$. So the company can not expect to stay in business unless earning (on average) more than is taken away by claims.

Drug testing

A hospital wishes to determine which of two drugs is more effective in treating a certain disease. Independently, they take pairs of patients and give the first one in the pair drug 1 and the second one in the pair drug 2. Each drug either cures or does not cure, and the outcome of the i^{th} pair is given by (A_i, B_i) where $A_i = 1$ if drug 1 cured the patient taking drug 1, 0 if not, and $B_i = 1$ if drug 2 cured the patient taking drug 2, 0 if not.

$$Z_n = \sum_{j=1}^{n}(A_j - B_j),\ Z_0 = 0,$$

then denotes the net difference (between the two drugs) in the number of cures out of n trials (pairs). If $Z_n > 0$ then drug 1 is ahead, whereas if $Z_n < 0$ then drug 2 is ahead. If $Z_n = 0$ then neither is ahead.

To determine which drug is better (we assume that one of them is strictly better), the hospital chooses and fixes an integer $M > 0$ (large) and waits until the first trial n until $Z_n = M$ or $Z_n = -M$, whichever happens first. If M is reached first, then they will conclude that drug 1 is better, whereas if $-M$ is reached first, then they will conclude that drug 2 is better. Clearly if M is sufficiently large, we expect this to be a good way of deciding which of the two drugs is more effective. But there is still the possibility that a mistake is made, and it is in our interest to explore this further.

Let $r_1 = P(A = 1),\ r_2 = P(B = 1)$ denote the apriori unknown individual "cure" probabilities, and assume that $\{A_i\}$ and $\{B_i\}$ form independent sequences each of which is an i.i.d. Bernoulli sequence with success probabilities r_1 and r_2 repectively. One can interpret r_1 as the long run proportion of patients cured by drug 1, and r_2 as the long run proportion of patients cured by drug 2. The hospital's objective is to determine which is larger, r_1 or r_2.

Now assume that $r_1 > r_2$ are known values, and let us compute the probability that the above method would wrongly assert that $r_2 > r_1$. $\{Z_n\}$ forms a random walk in which the increments $A_j - B_j$ take on the three values 1,$-$1, 0 with probabilities $P(A_j - B_j = 1) = P(A_j = 1, B_j = 0) = r_1(1 - r_2),\ P(A_j - B_j = -1) = r_2(1 - r_1),\ P(A_j - B_j = 0) = P(A_j = 1, B_j = 1) + P(A_j = 0, B_j = 0) = r_1r_2 + (1 - r_1)(1 - r_2)$.

We want then to compute the probability that Z_n first reaches level $-M$ before reaching level M under the above probabilities. Note that this probability can be cast in the framework of the gambler's ruin problem: Letting $i = M$, and $N = 2M$, we want to compute the probability $1 - P_i$ that the gambler is ruined, that is, the probability that Z_n goes down by M before going up by M. The only difference is the possibility here that no money is earned or lost on a given "gamble", e.g., $P(A_j - B_j = 0) > 0$. But since whenever $A_j - B_j = 0$, Z_n does not change value, we need only consider those gambles $A_j - B_j$ which yield 1 or -1. In other words, define new i.i.d. increments $\{\Delta_n\}$, and a corresponding new random walk $X_n = \Delta_1 + \cdots + \Delta_n$, in which

$$P(\Delta = 1) = p,\ P(\Delta = -1) = 1 - p,$$

where

$$p \ = \ P(A_j - B_j = 1 | A_j - B_j \neq 0)$$

$$
\begin{aligned}
&= \frac{P(A_j - B_j = 1)}{P(A_j - B_j = 1) + P(A_j - B_j = -1)} \\
&= \frac{r_1(1-r_2)}{r_1(1-r_2) + r_2(1-r_1)}. \qquad (6)
\end{aligned}
$$

Δ has the conditional distribution of an old increment $A_i - B_i$ given that $A_i - B_i \neq 0$.

This then places us exactly in the gambler's ruin framework, and we wish to compute (using the new chain $\{X_n\}$) $1 - P_i$ when $N = 2M$, and $i = M$. (3) the yields (after some algebra) the solution

$$1 - P_M = \frac{1}{1 + (p/q)^M},$$

where p is given by (6), and $q = 1 - p$. Clearly, by choosing M large enough, this probability can be made arbitrarily small.

1.3 Random walk hitting probabilities

Let $a > 0$ and $b > 0$ be integers, and let $R_n = \Delta_1 + \cdots + \Delta_n,\ n \geq 1,\ R_0 = 0$ denote a simple random walk initially at the origin. Let

$$p(a) = P(R_n \text{ hits level } a \text{ before hitting level } -b).$$

By letting $i = b$, and $N = a + b$, we can equivalently imagine a gambler who starts with $i = b$ and wishes to reach $N = a + b$ before going broke. So we can compute $p(a)$ by casting the problem into the framework of the gamblers ruin problem: $p(a) = P_i$ where $N = a + b$, $i = b$. Thus

$$p(a) = \begin{cases} \frac{1-(\frac{q}{p})^b}{1-(\frac{q}{p})^{a+b}}, & \text{if } p \neq q; \\ \frac{b}{a+b}, & \text{if } p = q = 0.5. \end{cases} \qquad (7)$$

Examples

1. Ellen bought a share of stock for \$10, and it is believed that the stock price moves (day by day) as a simple random walk with $p = 0.55$. What is the probability that Ellen's stock reaches the high value of \$15 before the low value of \$5?

   SOLUTION

   We want "the probability that the stock goes up by 5 before going down by 5." This is equivalent to starting the random walk at 0 with $a = 5$ and $b = 5$, and computing $p(a)$.

   $$p(a) = \frac{1-(\frac{q}{p})^b}{1-(\frac{q}{p})^{a+b}} = \frac{1-(0.82)^5}{1-(0.82)^{10}} = 0.73$$

2. What is the probability that Ellen will become infinitely rich?

   SOLUTION

   Here we equivalently want to know the probability that a gambler starting with $i = 10$ becomes infinitely rich before going broke. Just like Example 2 on Page 3:

   $1 - (q/p)^i = 1 - (0.82)^{10} \approx 1 - 0.14 = 0.86.$

1.4 Maximums and minimums of the simple random walk

Formula (7) can also be made use of for computing the probability that a random walk, starting initially at 0, will ever hit level a, for any given positive integer $a \geq 1$. All one has to do is keep a fixed while taking the limit as $b \to \infty$. The result depends on wether $p < 0.50$ or $p \geq 0.50$. A little thought reveals that we can state this problem as computing the tail $P(M \geq a)$ where $M \stackrel{\text{def}}{=} \max\{R_n : n \geq 0\}$ is a non-negative random variable.

Proposition 1.1 *When $p < 0.50$,*

$$P(M \geq a) = (p/q)^a, \;\; a \geq 0;$$

we conclude that M has a geometric distribution with "success" probability $\hat{p} = 1 - (p/q)$: $P(M = k) = (1 - \hat{p})^k \hat{p}, \;\; k \geq 0$, and thus (for example)

$$E(M) = \frac{(1 - \hat{p})}{\hat{p}} = \frac{(p/q)}{1 - (p/q)}.$$

If $p \geq 0.50$, then $P(M \geq a) = 1, \;\; a \geq 0$.

Proof : Taking the limit in (7) as $b \to \infty$ yields the result by considering the two cases $p < 0.5$ or $p \geq 0.5$: If $p < 0.5$, then $(q/p) > 1$ and so both $(q/p)^b$ and $(q/p)^{a+b}$ tend to ∞ as $b \to \infty$. But before taking the limit, multiply both numerator and denominator by $(q/p)^{-b} = (p/q)^b$, yielding

$$p(a) = \frac{(p/q)^b - 1}{(p/q)^b - (q/p)^a}.$$

Since $(p/q)^b \to 0$ as $b \to \infty$, the result follows.

If $p > 0.5$, then $(q/p) < 1$ and so both $(q/p)^b$ and $(q/p)^{a+b}$ tend to 0 as $b \to \infty$ yielding the limit in (7) as 1. If $p = 0.5$, then $p(a) = b/(b + a) \to 1$ as $b \to \infty$. ∎

If $p < 0.5$, then $E(\Delta) < 0$, and if $p > 0.5$, then $E(\Delta) > 0$; so Proposition 1.1 is consistent with the fact that any random walk with $E(\Delta) < 0$ (called the *negative drift* case) satisfies $\lim_{n\to\infty} X_n = -\infty$, wp1. and any random walk with $E(\Delta) > 0$ (called the *positive drift* case) satisfies $\lim_{n\to\infty} X_n = +\infty$, wp1. [2]

But furthermore we learn that when $p < 0.5$, although wp1 the chain drifts off to $-\infty$, it first reaches a finite maximum M before doing so, and this rv M has a geometric distribution.

Finally Proposition 1.1 also offers us a proof that when $p = 0.5$, the *symmetric* case, the random walk will wp1 hit any positive value, $P(M \geq a) = 1$.

By symmetry, we also obtain analogous results for the minimum, $m \stackrel{\text{def}}{=} \min\{X_n : n \geq 0\}$:

Corollary 1.1 *If $p > 0.5$, then*

$$P(m \leq -b) = (q/p)^b, \;\; b \geq 0.$$

In this case, the random walk drifts off to $+\infty$, but before doing so drops down to a finite minimum $m \leq 0$. Taking absolute values of m makes it non-negative and so we can express this

[2] From the strong law of large numbers, $\lim_{n\to\infty} \frac{X_n}{n} = E(\Delta)$, wp1, so $X_n \approx nE(\Delta) \to -\infty$ if $E(\Delta) < 0$ and $\to +\infty$ if $E(\Delta) > 0$.

result as $P(|m| \geq b) = (q/p)^b, \; b \geq 0$; $|m|$ *has a geometric distribution with "success" probability* $\hat{p} = 1 - (q/p)$*:* $P(M = k) = (1 - \hat{p})^k \hat{p}, \; k \geq 0$*.*

If $p \geq 0.5$*, then* $P(m \leq b) = 1, \; b \geq 0$*; the random walk will wp1 hit any negative value.*

Note that when $p < 0.5$, $P(M = 0) = 1 - (p/q) > 0$. This is because it is possible that the random walk will never enter the positive axis before drifting off to $-\infty$; $X_n \leq 0, \; n \geq 0$.

Recurrence of the simple symmetric random walk

Combining the results for both M and m when $p = 0.5$, $P(M \geq a) = 1 = P(m \leq -b) = 1$, we see that

> *The simple symmetric random walk, starting at the origin, will wp1 hit any value, positive or negative.*

Using the above, we know that the simple symmetric random walk will hit 1 eventually. But when it does, we can use the same result to conclude that it will go back and hit 0 eventually after that, because that is equivalent to starting at $X_0 = 0$ and waiting for the chain to hit -1, which also would happen eventually. But then yet again it must hit 1 again and so on (back and forth infinitely often), all by the same logic. *We conclude that the chain will, over and over again, return to state* 0 *wp1; it will do so infinitely often;* 0 *is a recurrent state for the simple symmetric random walk. Thus (since the chain is irreducible) all states are recurrent.*

Let $\tau = \min\{n \geq 1 : X_n = 0 \mid X_0 = 0\}$, the so-called *return time* to state 0. We just argued that τ is a proper random variable, that is, $P(\tau < \infty) = 1$. This means that if the chain starts in state 0, then, if we wait long enough, we will (wp1) see it return to state 0. What we will prove later is that $E(\tau) = \infty$; meaning that on average our wait is infinite. This implies that *the simple symmetric random walk forms a null recurrent Markov chain.*