IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

SOLUTIONS to Homework Assignment 10

More Brownian motion, due on Monday, August 20

Do the following exercises at the end of Chapter 10.

1. Exercise 10.8. Do a generalization of this problem in which the random walk goes up $\sigma\sqrt{\Delta t}$ with probability $(1/2)(1 + (\mu/\sigma)\sqrt{\Delta t})$ and otherwise goes down to $-\sigma\sqrt{\Delta t}$, where Δt is a small quantity (not a function of t). It is convenient to think of $\Delta t = 1/n$ and then let $n \to \infty$. The essential requirement is to show that the means and variances converge. The random walk has stationary independent increments, so too will the limit process.

Background: We give additional explanation: To rigorously show that the family of random walks converges to Brownian motion, we should apply a version of the central limit theorem. To get non-zero drift in the suggested manner, we actually want to apply a version of the central limit theorem for an array of random variables. For each $n \geq 1$, we have m_n random variables $X_{n,i}$ for $1 \leq i \leq m_n$, where the m_n random variables are i.i.d. for each n. We then form the sums $S_n \equiv X_{n,1} + \cdots X_{n,m_n}$. It usually suffices to let $m_n = n$ for each n. We then want to conclude that S_n converges in distribution to $N(\mu, \sigma^2)$ as $n \to \infty$, where $N(\mu, \sigma^2)$ denotes a random variable that is normally distributed with mean μ and variance σ^2. For simple random walks like we are considering, the following theorem applies:

Theorem 0.1 (central limit theorem for arrays of bounded random variables) *Suppose that $\{X_{n,i} : 1 \leq i \leq m_n\}$ are i.i.d random variables for each n (which permits the distribution to depend upon n). Moreover, suppose that $X_{n,i}$ are bounded, i.e., $P(X_{n,i} \leq c_n) = 1$ for all n, where the bounds are asymptotically negligible as $n \to \infty$, i.e., where $c_n \to 0$ as $n \to \infty$. If*

$$E[S_n] \to \mu \quad \text{and} \quad Var(S_n) \to \sigma^2 \quad \text{as} \quad n \to \infty ,$$

then $S_n \Rightarrow N(\mu, \sigma^2)$ as $n \to \infty$, where $\Rightarrow$ denotes convergence in distribution.

Note that the usual single-sequence setting with mean-zero random variables applies here by setting $X_{n,i} = X_i/\sqrt{n}$, because the spatial normalization is contained in the $X_{n,i}$. This theorem is a special case of a more general theorem, which allows the random variables to be unbounded and to have different distributions as i changes for each n. Such a theorem is Theorem 5.15 on page 93 of O. Kallenberg, *Foundations of Modern Probability*, second edition, Springer, 2002. By virtue of this theorem, we can establish convergence in distribution of the random walk to Brownian motion for each time point t by showing convergence of means and variances, as in the condition above.

To proceed in this framework, fix some small increment of time Δt, as in the problem statement. Then let $n = 1/\Delta t$. (Actually the t in this notation may be confusing, and is not needed, but we leave it there. We point out that n is not a function of t.)

Let the approximating random walk at time t be the random walk value after $t/\Delta t = nt$ steps, i.e.,

$$S_n(t) \equiv \sum_{i=1}^{nt} X_{n,i} \ ,$$

where we understand nt to be an integer (the greatest integer less than nt, often written as $\lfloor nt \rfloor$) and $X_{n,i}$ is as in the problem formulation (matching n to $1/\Delta t$); i.e.,

$$P(X_{n,i} = \sigma/\sqrt{n} = \sigma\sqrt{\Delta t}) = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right) = 1 - P(X_{n,i} = -\sigma/\sqrt{n} = -\sigma\sqrt{\Delta t}) \ .$$

To apply our theorem above, and thus complete this exercise, we need to show that

$$E[S_n(t)] \to \mu t \quad \text{and} \quad Var(S_n(t)) \to \sigma^2 t \quad \text{as} \quad n \to \infty \ .$$

That shows that $S_n(t) \Rightarrow N(\mu t, \sigma^2 t)$ as $n \to \infty$ for each t, but the Brownian motion $\mu t + \sigma B(t) \stackrel{\mathrm{d}}{=} N(\mu t, \sigma^2 t)$. We note that the random walks also have stationary independent increments, so that property will be inherited by the limit process. Hence the limit process must actually be Brownian motion, with the specified drift and diffusion coefficient.

answers: (a) As stated above, it suffices to show that

$$E[S_n(t)] \to \mu t \quad \text{and} \quad Var(S_n(t)) \to \sigma^2 t \quad \text{as} \quad n \to \infty \ .$$

Note that the means work out directly; we need no limit:

$$\begin{aligned} E[S_n(t)] &= ntE[X_{n,i}] \\ &= nt(\frac{1}{2}\left(1 + \frac{\mu}{\sigma\sqrt{n}}\right)(\sigma/\sqrt{n}) + \frac{1}{2}\left(1 - \frac{\mu}{\sigma\sqrt{n}}\right)(-\sigma/\sqrt{n})) \\ &= nt(\mu/n) = \mu t \ . \end{aligned} \tag{1}$$

The variances are a little more complicated. First notice that

$$Var(S_n(t)) = ntVar[X_{n,i}] \ .$$

What we want to show, then, is that $Var(X_{n,i}) = \sigma^2/n + o(1/n)$, where $o(1/n)$ is a quantity that is asymptotically negligible compared to $1/n$. It is easy to directly calculate the second moment of $X_{n,i}$:

$$E[(X_{n,i})^2] = \sigma^2/n \ .$$

But then the variance is

$$Var(X_{n,i}) = E[(X_{n,i})^2] - (E[X_{n,i}])^2 = \sigma^2/n - (\mu/n)^2 \ .$$

Hence,

$$Var(S_n(t)) = ntVar(X_{n,i}) = nt(\sigma^2/n - (\mu/n)^2)) = \sigma^2 t - \frac{\mu^2 t}{n} \to \sigma^2 t \ ,$$

as claimed.

(b) Hint: Specifically, apply (4.14) on page 219. To do so, use the following basic limit:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x \ . \tag{2}$$

Let p be the probability of going up A before going down B in the BM. Let p_n be the probability of going up to $\frac{\sqrt{n}(A+B)}{\sigma}$ before going down to 0 starting in state $\frac{\sqrt{n}B}{\sigma}$ in the gambler's ruin problem. Then we should have $p_n \to p$ as $n \to \infty$. But $p_n = a_n/b_n$, where

$$a_n = 1 - \left(\frac{1 - \frac{\mu}{\sigma\sqrt{n}}}{1 + \frac{\mu}{\sigma\sqrt{n}}} \right)^{B\sqrt{n}/\sigma}$$

and

$$b_n = 1 - \left(\frac{1 - \frac{\mu}{\sigma\sqrt{n}}}{1 + \frac{\mu}{\sigma\sqrt{n}}} \right)^{(A+B)\sqrt{n}/\sigma} ,$$

but, by (2),

$$\left(1 - \frac{\mu}{\sigma\sqrt{n}}\right)^{\sqrt{n}} \to e^{-\mu/\sigma}$$

and

$$\left(1 + \frac{\mu}{\sigma\sqrt{n}}\right)^{\sqrt{n}} \to e^{+\mu/\sigma}$$

as $n \to \infty$. Hence,

$$p_n \to \frac{1 - \frac{e^{-B\mu/\sigma^2}}{e^{+B\mu/\sigma^2}}}{1 - \frac{e^{-(A+B)\mu/\sigma^2}}{e^{+(A+B)\mu/\sigma^2}}} = \frac{1 - e^{-2B\mu/\sigma^2}}{1 - e^{-2(A+B)\mu/\sigma^2}} .$$

Remark: It is possible to derive this result directly for Brownian motion with drift by applying martingales; see Exercise 10.22.

2. Exercise 10.11. Hint: Apply Exercise 10.8 after taking logarithms. To do so, we want to use a version of the Continuous Mapping Theorem: If $Y_n \Rightarrow Y$ as $n \to \infty$ and if f is a continuous function, then $f(Y_n) \Rightarrow f(Y)$ as $n \to \infty$, where $\Rightarrow$ denotes convergence in distribution.

answer: Let Z_n be the value of the process at step n. To make the connection to Exercise 10.8, let $h = \Delta t = 1/n$ After taking logarithms, we get $\ln Z_n$ going up to $\sigma\sqrt{h}$ with probability p, and down to $-\sigma\sqrt{h}$ with probability $1-p$, where p is the same as in Exercise 10.8. Thus, $\ln(Z_n(t))$ is just $S_n(t)$ as defined in the problem above. Since, $S_n(t) \Rightarrow \mu t + \sigma B(t)$ as $n \to \infty$, $\ln(Z_n(t)) \Rightarrow \mu t + \sigma B(t)$ as $n \to \infty$. Since the exponential is a continuous function, we have

$$Z_n(t) = e^{\ln(Z_n(t))} \Rightarrow e^{\mu t + \sigma B(t)} \quad \text{as} \quad n \to \infty ,$$

but the limit is geometric Brownian motion.

3. Exercise 10.15 (a). Hint: Apply formula (10.12).

answer: With reference to formula (10.12), the initial stock price is $x_0 = 100$, the strike price is $K = 100$, the interest rate is $\alpha = 0.05$ (with continuous compounding), the volatility is $\sigma = 1.0$ (exceptionally high, not realistic), and the expiration time is $t = 10$. The drift is given as $\mu = 2$, but that is irrelevant; it plays no role. We use instead the risk neutral version with $\mu = \alpha - \sigma^2/2$, as given on line 4 of page 640.

Then

$$b = \frac{(0.05)(10) - (1)(10)/2 - \ln(100/100)}{1\sqrt{10}} = \frac{-4.5}{\sqrt{10}} = \frac{-4.5}{3.162} = -1.423$$

and the cost itself (using Φ instead of ϕ for the standard normal cdf) is

$$c = 100\Phi(\sqrt{10} - 1.423) - 100e^{-0.5}\Phi(-1.423) = 100\Phi(1.739) - 100e^{-0.5}(1 - \Phi(1.423)) = 91.2$$

4. Exercise 10.16. Hint: Apply a generalization of formula (3.3) on page 106.

answer: The martingale property says that

$$E[Y(t)|Y(u), 0 \le u \le s] = Y(s)$$

for $0 \le s < t$. Taking expectations of both sides gives the conclusion:

$$E[Y(t)] = E[Y(s)] \quad \text{for all} \quad s, \quad 0 \le s < t \ .$$

In particular, we apply the result with $s = 0$.

5. Exercise 10.16. Hint: Compare to Example 2 in the Sigman martingale notes. Apply Proposition 1.1 there.

answer: (a) Show that Y is a martingale (MG) with respect to B. That implies that Y is an MG with respect to Y; see Proposition 1.1 of the Sigman notes

$$\begin{aligned} E[B(t)^2 - t|B(u), 0 \le u \le s] &= E[(B(s) + B(t) - B(s))^2 - t|B(u), 0 \le u \le s] \\ &= B(s)^2 + 2B(s)E[B(t) - B(s)|B(u), 0 \le u \le s] \\ &\qquad -E[(B(t) - B(s))^2|B(u), 0 \le u \le s] - t \\ &= B(s)^2 + 2B(s)E[B(t) - B(s)] - E[(B(t) - B(s))^2] - t \\ &= B(s)^2 + 2B(s)0 - (t - s) - t \\ &= B(s)^2 - s \end{aligned}$$

We should also check that the expectations are finite:

$$E[|B(t)^2 - t|] \le E[B(t)^2 + t \le 2t \ .$$

(b) By Exercise 10.16, $E[Y(t)] = E[Y(0)] = E[B(0)^2 - 0] = 0 - 0 = 0$.

6. Exercise 10.21. But simply assume that the conditions of the Optional Stopping Theorem are satisfied.

answer: Given the MG stopping Theorem,

$$E[B(T) = E[B(0)] = 0 \ ,$$

but $B(T) = (x - \mu T)/\sigma$, so that

$$E[(x - \mu T)/\sigma] = 0 \quad \text{or} \quad E[T] = x/\mu \ .$$

7. Exercise 10.25. Hint: Read the first two pages of Section 10.5.

answer: The means equal 0.

$$Var\left(\int_0^1 t \, dB(t)\right) = \int_0^1 t^2 \, dt = \frac{1}{3}$$

and

$$Var\left(\int_0^1 t^2\, dB(t)\right) = \int_0^1 t^4\, dt = \frac{1}{5}\ .$$

8. the heat equation

Let $f(t,x)$ be a function of two variables representing the density of standard Brownian motion at time t, i.e., the normal density with mean 0 and variance t:

$$f(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

Show that f satisfies the heat equation, i.e., the partial differential equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\ .$$

answer: Differentiating gives:

$$\frac{\partial f}{\partial t} = \frac{-1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{t^{3/2}}e^{-x^2/2t} + \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\frac{x^2}{2t^2}\ ,$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\left(\frac{-x}{t}\right)$$

and then

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\left(\frac{x^2}{t^2}\right) + \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\left(\frac{-1}{t}\right)\ ,$$

which implies the desired conclusion.