IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

Homework Assignment 11, due on Monday, August 20

stochastic calculus, Black-Scholes and martingales

1. Pricing The Squared Derivative

Consider a stock whose price follows geometric Brownian motion (GBM), according to either a specification as the exponential of BM,

$$S(t) = S(0)e^{X(t)} = S(0)e^{\nu t + \sigma B(t)}, \quad t \geq 0 \ , \tag{1}$$

or a specification as a stochastic differential equation (SDE),

$$dS(t) = \mu S dt + \sigma S dB \ . \tag{2}$$

For these to be consistent, we need

$$\mu = \nu + \frac{\sigma^2}{2} \quad \text{or} \quad \nu = \mu - \frac{\sigma^2}{2} \ . \tag{3}$$

But the parameter σ is consistent, so no adjustment of it is needed. For more discussion, see Example 4.1 in the class lecture notes of August 13.

(a) Given that S satisfies the SDE in (2), apply Ito's lemma to find the associated SDE satisfied by the stochastic process $\{S(t)^2 : t \geq 0\}$. Show that this SDE is a GBM SDE too with new coefficients μ and σ.

(b) Show how the conclusion in part (a) can also be derived from the alternative representation in (1).

(c) Now consider a (financial) derivative that pays off $S(T)^2$ at a fixed expiration time T, given that the stock price itself is then $S(T)$. Assume that there is a fixed interest rate r. Price this squared derivative; i.e., find the unique arbitrage-free price. Hint: Recall that the arbitrage-free price is the discounted expected value with respect to the risk-neutral GBM, where the risk-neutral GBM is obtained from any given GBM by setting $\mu = r$ or, equivalently, by setting $\nu = r - \sigma^2/2$.

(d) Use your answer in part (c) to get an expression for $f(x,t)$, defined to be the price of this derivative at time t if $S(t) = x$, for all possible $x \geq 0$ and t, $0 \leq t \leq T$.

(e) Verify that this state-dependent and time-dependent price $f(x,t)$ satisfies the Black-Scholes partial differential equation:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} rx + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} = rf \ . \tag{4}$$

2. Pricing A Digital Call Option

As in the previous problem, consider a stock whose price follows geometric Brownian motion (GBM), according to either a specification as the exponential of BM, as in (1), or a specification as a stochastic differential equation (SDE), as in (2). For these to be consistent, we need ν and μ to be related as in (3). Now consider a derivative, called the *digital call option*, that pays off \$1 if the stock price exceeds the strike price K at the fixed expiration time T, and pays off nothing otherwise. As before, assume that there is a fixed interest rate r.

(a) Price this digital call option; i.e., find the unique arbitrage-free price. Hint: Recall the hint in Problem 1 (c).

(b) Relate the price of this digital call option just derived to one of the two terms in the Black-Scholes formula for the arbitrage-free price of a European call option, as given in (10.12) on page 641 of Ross.

3. Poisson Martingales

(a) Let $N \equiv \{N(t) : t \geq 0\}$ be a Poisson counting process with intensity (rate) λ. Show that the stochastic process $\{N(t) - \lambda t : t \geq 0\}$ is a martingale, with respect to $N \equiv \{N(t) : t \geq 0\}$.

(b) Show that the stochastic process $\{M(t)^2 - \lambda t : t \geq 0\}$ is also a martingale, with respect to $N \equiv \{N(t) : t \geq 0\}$, where $M(t) = N(t) - \lambda t$, $t \geq 0$.

(c) Let T_7 be the first time that the Poisson process reaches the level 7. Use the martingales in parts (a) and (b) plus the Optional Stopping Theorem (without checking the technical conditions) to calculate the first two moments of T_7.

(d) Check your answer in part (c) by representing T_7 as the sum of 7 i.i.d. random variables.

4. Geometric Brownian Motion Plus Negative Jumps

Let $S(t)$ be a stock price at time t. Let $Y(t) = \ln(S(t)/S(0))$, be the logarithm of the ratio of the stock price at time t to its price at time 0. Suppose that $Y(t) = X(t) - bD(t)$, where $X(t) = \nu t + \sigma B(t)$, where $B \equiv \{B(t) : t \geq 0\}$ is standard Browmian motion, and $D \equiv \{D(t) : t \geq 0\}$ is a Poisson process independent of $\{X(t) : t \geq 0\}$ with rate λ. Notice that the process Y has negative jumps of size b at random times.

(a) Does the stochastic process Y have independent increments?

(b) Does the stochastic process Y have stationary increments?

(c) Is the stochastic process $\{Y(t) - ct : t \geq 0\}$ a martingale relative to the stochastic process $\{(X(t), D(t)) : t \geq 0\}$ for some value of c? (Note that the stochastic process $\{(X(t), D(t)) : t \geq 0\}$ contains all the information of both processes being subtracted.)

(d) What is the conditional mean of $Y(t+s)$ given that $Y(s) = y$?

(e) What is the conditional variance of $Y(t+s)$ given that $Y(s) = y$?

5. integration by parts

Suppose that f and g are two continuously differentible functions, i.e., functions with continuous derivatives. Then the integration by parts formula is

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx \ . \qquad (5)$$

(a) Show this formula is justified by verifying the following relation for sums

$$\sum_{i=1}^{n} a_i(b_{i+1} - b_i) = a_n b_{n+1} - a_1 b_1 - \sum_{i=2}^{n} b_i(a_i - a_{i-1}) \tag{6}$$

These two sums can be regarded as the approximating Riemann sums in the definition of the Riemann integrals, yielding

$$\sum_{i=na+1}^{nb} f(i/n)(g((i+1)/n - g(i/n))$$
$$= f(b)g(b + (1/n)) - f(a + (1/n)g(a + (1/n)) - \sum_{i=na+2}^{nb} g(i/n)(f(i/n) - f((i-1)/n)) \tag{7}$$

As $n \to \infty$, formula (7) approaches (5).

(b) Let $B \equiv \{B(t) : t \geq 0\}$ be standard Browmian motion and let f be a real-valued function having a continuous derivative f' on $[a, b]$. Suppose that we define the stochastic integral

$$\int_a^b f(x)\, dB(x) \tag{8}$$

by an appropriate limit of the approximating sums

$$\sum_{i=na+1}^{nb} f(i/n)(B((i+1)/n - B(i/n)) \tag{9}$$

as $n \to \infty$, just as in (7). Use (6) and the continuity of the paths of Brownian motion to justify the formula

$$\int_a^b f(x)\, dB(x) = f(b)B(b) - f(a)B(a) - \int_a^b B(x)f'(x)\, dx \tag{10}$$

(c) Find the full distribution of the discounted present value of a Brownian income stream, with discount rate r, i.e., of

$$D(r) \equiv \int_0^\infty e^{-rt}\, dB(t) \tag{11}$$

Hint: Apply Problem 1 on homework 9.