IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

SOLUTIONS to Homework Assignment 11

stochastic calculus, Black-Scholes and martingales

1. Pricing The Squared Derivative

Consider a stock whose price follows geometric Brownian motion (GBM), according to either a specification as the exponential of BM,

$$S(t) = S(0)e^{X(t)} = S(0)e^{\nu t + \sigma B(t)}, \quad t \geq 0 \; , \tag{1}$$

or a specification as a stochastic differential equation (SDE),

$$dS(t) = \mu S dt + \sigma S dB \; . \tag{2}$$

For these to be consistent, we need

$$\mu = \nu + \frac{\sigma^2}{2} \quad \text{or} \quad \nu = \mu - \frac{\sigma^2}{2} \; . \tag{3}$$

But the parameter σ is consistent, so no adjustment of it is needed. For more discussion, see Example 4.1 in the class lecture notes of August 13.

(a) Given that S satisfies the SDE in (2), apply Ito's lemma to find the associated SDE satisfied by the stochastic process $\{S(t)^2 : t \geq 0\}$. Show that this SDE is a GBM SDE too with new coefficients μ and σ.

We apply Ito's lemma with the function $f(x,t) = x^2$. We have the partial derivatives $f_x(x,t) = 2x$, $f_{x,x}(x,t) = 2$ and $f_t(x,t) = 0$. Hence, Ito's lemma gives us for $Y(t) = S(t)^2$:

$$\begin{aligned}
dY &= (f_t + a f_x + \frac{1}{2} b^2 f_{x,x})dt + b f_x dB \\
&= (f_t + \mu S f_x + \frac{1}{2}\sigma^2 S^2 f_{x,x})dt + \sigma S f_x dB \\
&= (0 + \mu S(2S) + \frac{1}{2}\sigma^2 S^2 (2))dt + \sigma S(2S) dB \\
&= (2\mu S^2 + \sigma^2 S^2)dt + 2\sigma S^2 dB \\
&= ((2\mu + \sigma^2)S^2)dt + (2\sigma)S^2 dB \\
&= ((2\mu + \sigma^2)Y)dt + (2\sigma)Y dB,
\end{aligned}$$

so that Y too satisfies the GBM SDE but with new coefficients, with μ replaced by $2\mu + \sigma^2$ and σ replaced by 2σ.

(b) Show how the conclusion in part (a) can also be derived from the alternative representation in (1).

From (1),

$$Y(t) = S(t)^2 = S(0)^2 e^{2X(t)} = S(0)^2 e^{2\nu t + 2\sigma B(t)} ,$$

which is again of the same form, with $S(0)$ replaced by $S(0)^2$, ν replaced by 2ν and σ replaced by 2σ. By (3), that makes the new transformed value of μ equal to $2\nu + (2\sigma)^2/2 = 2\nu + 2\sigma^2$. In terms of the original μ, this transformed μ is then equal to $2\mu - \sigma^2 + 2\sigma^2 = 2\mu + \sigma^2$, which is precisely what we derived in part (a).

(c) Now consider a (financial) derivative that pays off $S(T)^2$ at a fixed expiration time T, given that the stock price itself is then $S(T)$. Assume that there is a fixed interest rate r. Price this squared derivative; i.e., find the unique arbitrage-free price. Hint: Recall that the arbitrage-free price is the discounted expected value with respect to the risk-neutral GBM, where the risk-neutral GBM is obtained from any given GBM by setting $\mu = r$ or, equivalently, by setting $\nu = r - \sigma^2/2$.

The initial arbitrage-free price at time 0 is

$$C(0) = e^{-rT} E^*[C(T)] = e^{-rT} E^*[S(T)^2] , \tag{4}$$

where E^* is the expectation with respect to the risk-neutral GBM (with respect to r). In general, $S(t)^2 = S(0)^2 e^{2X(t)} = S(0)^2 e^{2\nu t + 2\sigma B(t)}$, so that, using the mgf of the normal distribution on p. 67, we get

$$\begin{aligned} E[S(t)^2] &= S(0)^2 E[e^{2X(t)}] = S(0)^2 E[e^{2\nu t + 2\sigma B(t)}] \\ &= S(0)^2 e^{2\nu t} E[e^{2\sigma B(t)}] \\ &= S(0)^2 e^{2\nu t} e^{(4\sigma^2 t)/2} \\ &= S(0)^2 e^{2\nu t} e^{2\sigma^2 t} , \end{aligned}$$

so that, upon letting $\nu = r - \sigma^2/2$

$$C(0) = e^{-rT} E^*[S(T)^2] = e^{-rT} S(0)^2 e^{2rT + \sigma^2 T} = S(0)^2 e^{(r+\sigma^2)T} . \tag{5}$$

(d) Use your answer in part (c) to get an expression for $f(x,t)$, defined to be the price of this derivative at time t if $S(t) = x$, for all possible $x \geq 0$ and t, $0 \leq t \leq T$.

Since the original initial price $S(0)$ and expiration time T were specified as variables, it is immediate that

$$f(x,t) = x^2 e^{(r+\sigma^2)(T-t)} .$$

(e) Verify that this state-dependent and time-dependent price $f(x,t)$ satisfies the Black-Scholes partial differential equation:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} rx + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} = rf . \tag{6}$$

The partial derivatives with respect to x satisfy:

$$f_x = 2xe^{(r+\sigma^2)(T-t)} \quad \text{and} \quad f_{x,x} = 2e^{(r+\sigma^2)(T-t)} ,$$

while the partial derivative with respect to t satisfies

$$f_t = x^2(-(r+\sigma^2))e^{(r+\sigma^2)(T-t)} .$$

It is then easy to verify that the equation is satisfied.

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2. Pricing A Digital Call Option

As in the previous problem, consider a stock whose price follows geometric Brownian motion (GBM), according to either a specification as the exponential of BM, as in (1), or a specification as a stochastic differential equation (SDE), as in (2). For these to be consistent, we need ν and μ to be related as in (3). Now consider a derivative, called the *digital call option*, that pays off \$1 if the stock price exceeds the strike price K at the fixed expiration time T, and pays off nothing otherwise. As before, assume that there is a fixed interest rate r.

(a) Price this digital call option; i.e., find the unique arbitrage-free price. Hint: Recall the hint in Problem 1 (c).

The initial arbitrage-free price at time 0 is

$$C(0) = e^{-rT}E^*[C(T)] = e^{-rT}E^*[1_{S(T)>K}] = e^{-rT}P(S^*(T) > K) ,$$

where E^* is the expectation with respect to the risk-neutral GBM (with respect to r), $S^*(T)$ is the associated price of the risk-neutral GBM at time T, and 1_A is the indicator function ($1_A = 1$ on the event A and equals 0 otherwise). But

$$P(S^*(T) > K) = P(X^*(T) > \ln(K/S(0)) ,$$

where $X^*(T) \stackrel{\mathrm{d}}{=} N((r-\sigma^2/2)T, \sigma^2 T)$ after adjusting to the risk neutral drift. Thus

$$\begin{aligned} P(X^*(T) > \ln(K/S(0)) &= P\left(\frac{X^*(T) - (r-\sigma^2/2)T}{\sqrt{\sigma^2 T}} > \frac{\ln(K/S(0)) - (r-\sigma^2/2)T}{\sqrt{\sigma^2 T}}\right) \\ &= P\left(N(0,1) > \frac{\ln(K/S(0)) - (r-\sigma^2/2)T}{\sqrt{\sigma^2 T}}\right) \\ &= P\left(N(0,1) \le \frac{-\ln(K/S(0)) + (r-\sigma^2/2)T}{\sqrt{\sigma^2 T}}\right) \\ &= \Phi\left(\frac{-\ln(K/S(0)) + (r-\sigma^2/2)T}{\sqrt{\sigma^2 T}}\right) \\ &= \Phi(c) \qquad (7) \end{aligned}$$

where Φ is the standard normal cdf and

$$c \equiv \frac{-\ln(K/S(0)) + (r-\sigma^2/2)T}{\sqrt{\sigma^2 T}} .$$

Finally, we have

$$C(0) = e^{-rT} E^*[C(T)] = e^{-rT} P(S^*(T) > K) = e^{-rT} \Phi(c)$$

for c just derived.

(b) Relate the price of this digital call option just derived to one of the two terms in the Black-Scholes formula for the arbitrage-free price of a European call option, as given in (10.12) on page 641 of Ross.

This term shows up in the second piece of the Black-Scholes pricing formula for a standard call option. The second term in (10.12) is precisely K times the price of the digital call option. Thus that term can be viewed as the price of K such digital options.

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3. Poisson Martingales

(a) Let $N \equiv \{N(t) : t \geq 0\}$ be a Poisson counting process with intensity (rate) λ. Show that the stochastic process $\{N(t) - \lambda t : t \geq 0\}$ is a martingale, with respect to $N \equiv \{N(t) : t \geq 0\}$.

The reasoning is essentially the same as for BM because the Poisson process also has stationary and independent increments. Note that

$$\begin{aligned}
E[N(t+u) - \lambda(t+u)|N(s), 0 \leq s \leq t] &= E[N(t) + N(t+u) - N(t) - \lambda(t+u)|N(s), 0 \leq s \leq t] \\
&= E[N(t)|N(s), 0 \leq s \leq t] \\
&\qquad + E[N(t+u) - N(t)|N(s), 0 \leq s \leq t] - \lambda(t+u) \\
&= N(t) + E[N(t+u) - N(t)] - \lambda(t+u) \\
&= N(t) + \lambda u - \lambda(t+u) \\
&= N(t) - \lambda t \ ,
\end{aligned}$$

as we wanted to show.

(b) Show that the stochastic process $\{M(t)^2 - \lambda t : t \geq 0\}$ is also a martingale, with respect to $N \equiv \{N(t) : t \geq 0\}$, where $M(t) = N(t) - \lambda t$, $t \geq 0$.

Note that

$$\begin{aligned}
&E[M^2(t+u) - \lambda(t+u)|N(s), 0 \leq s \leq t] = E[(N(t+u) - \lambda(t+u))^2 - \lambda(t+u)|N(s), 0 \leq s \leq t] \\
&\quad = E[N^2(t+u) - 2\lambda(t+u)N(t+u) + \lambda^2(t+u)^2|N(s), 0 \leq s \leq t] - \lambda(t+u) \\
&\quad = E[N^2(t+u)|N(s), 0 \leq s \leq t] - 2\lambda(t+u)E[N(t+u)|N(s), 0 \leq s \leq t] + \lambda^2(t+u)^2 - \lambda(t+u) \\
&\quad = E[(N(t+u) - N(t) + N(t))^2|N(s), 0 \leq s \leq t] \\
&\qquad -2\lambda(t+u)E[N(t+u) - N(t) + N(t)|N(s), 0 \leq s \leq t] + \lambda^2(t+u)^2 - \lambda(t+u) \\
&\quad = E[(N(t+u) - N(t))^2 + N^2(t) + 2N(t)(N(t+u) - N(t))|N(s), 0 \leq s \leq t] \\
&\qquad -2\lambda(t+u)\lambda u - 2\lambda(t+u)N(t) + \lambda^2(t^2 + 2ut + u^2) - \lambda(t+u) \\
&\quad = E[N^2(u)] + N^2(t) + 2N(t)\lambda u - 2\lambda^2 u(t+u) - 2\lambda N(t)(t+u) + \lambda^2 t^2 + 2\lambda^2 ut + \lambda^2 u^2 - \lambda t - \lambda u
\end{aligned}$$

$$= N^2(t) - 2\lambda t N(t) + \lambda^2 t^2 - \lambda t$$
$$= (N(t) - \lambda t)^2 - \lambda t$$
$$= M^2(t) - \lambda t\ ,$$

as we wanted to show.

(c) Let T_7 be the first time that the Poisson process reaches the level 7. Use the martingales in parts (a) and (b) plus the Optional Stopping Theorem (without checking the technical conditions) to calculate the first two moments of T_7.

Clearly $N(T_7) = 7$. By the OST applied to the MG in part (a), $E[N(T_7) - \lambda T_7] = E[N(0)] - \lambda 0 = 0$, so that $E[T_7] = 7/\lambda$. Now, applying the OST again with the MG from part (b), we have $E[M^2(T_7) - \lambda T_7] = E[M^2(0)] - \lambda 0 = 0$, so that $E[(N(T_7) - \lambda T_7)^2] = \lambda E[T_7]$ or

$$E[N(T_7)^2 - 2N(T_7)\lambda T_7 + \lambda^2 T_7^2] = \lambda E[T_7]$$

or

$$49 - 14\lambda E[T_7] + \lambda^2 E[T_7^2] = \lambda E[T_7]$$

or, substituting $E[T_7] = 7/\lambda$,

$$49 - 98 + \lambda^2 E[T_7^2] = 7 \quad \text{or} \quad E[T_7^2] = \frac{56}{\lambda^2}\ .$$

(d) Check your answer in part (c) by representing T_7 as the sum of 7 i.i.d. random variables.

Since N is a Poisson process, T_7 is the sum of 7 i.i.d. expoential random variables, each with mean $1/\lambda$. Hence the mean and variance are $E[T_7] = 7/\lambda$ and $Var(T_7) = 7(1/\lambda)^2 = 7/\lambda^2$. That makes the second moment $E[T_7^2] = Var(T_7) + E[T_7]^2 = (7/\lambda^2) + (49/\lambda^2) = 56/\lambda^2$, as deduced above.

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4. Geometric Brownian Motion Plus Negative Jumps

Let $S(t)$ be a stock price at time t. Let $Y(t) = \ln(S(t)/S(0))$, be the logarithm of the ratio of the stock price at time t to its price at time 0. Suppose that $Y(t) = X(t) - bD(t)$, where $X(t) = \nu t + \sigma B(t)$, where $B \equiv \{B(t) : t \geq 0\}$ is standard Browmian motion, and $D \equiv \{D(t) : t \geq 0\}$ is a Poisson process independent of $\{X(t) : t \geq 0\}$ with rate λ. Notice that the process Y has negative jumps of size b at random times.

(a) Does the stochastic process Y have independent increments?

Yes, both Brownian motion and the Poisson process have independent increments, and it is assumed that the two processes are mutually independent.

(b) Does the stochastic process Y have stationary increments?

Yes, both Brownian motion and the Poisson process have stationary increments too.

(c) Is the stochastic process $\{Y(t)-ct : t \ge 0\}$ a martingale relative to the stochastic process $\{(X(t), D(t)) : t \ge 0\}$ for some value of c? (Note that the stochastic process $\{(X(t), D(t)) : t \ge 0\}$ contains all the information of both processes being subtracted.)

Yes, the sum of two martingales with respect to the same stochastic process or filtration is again a martingale. That follows directly from the linearity of conditional expectations. The component martingales are $\{X(t) - \mu t : t \ge 0\}$ and $\{bD(t) - b\lambda t : t \ge 0\}$ We thus have $\{Y(t) - (\mu - b\lambda)t : t \ge 0\}$ a MG; i.e., $c = \mu - b\lambda$.

(d) What is the conditional mean of $Y(t+s)$ given that $Y(s) = y$?

Given that $Y(s) = y$, $Y(t+s)$ is distributed the same as $Y(t) + y$. The conditional mean is $\mu t - b\lambda t + y = y + (\mu - b\lambda)t$. Both the Poisson process and Brownian motion have means proportional to t.

(e) What is the conditional variance of $Y(t+s)$ given that $Y(s) = y$?

The variance is the sum of the variances: The conditional variance is $\sigma^2 t + b^2\lambda t = (\sigma^2 + b^2\lambda)t$. Both the Poisson process and Brownian motion have variances proportional to t. Recall that the variance of a Poisson random variable equals its mean.

5. integration by parts

Suppose that f and g are two continuously differentible functions, i.e., functions with continuous derivatives. Then the integration by parts formula is

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx \ . \tag{8}$$

(a) Show this formula is justified by verifying the following relation for sums

$$\sum_{i=1}^{n} a_i(b_{i+1} - b_i) = a_n b_{n+1} - a_1 b_1 - \sum_{i=2}^{n} b_i(a_i - a_{i-1}) \tag{9}$$

These two sums can be regarded as the approximating Riemann sums in the definition of the Riemann integrals, yielding

$$\begin{aligned} &\sum_{i=na+1}^{nb} f(i/n)(g((i+1)/n - g(i/n)) \\ &\quad = f(b)g(b+(1/n)) - f(a+(1/n)g(a+(1/n)) \\ &\quad\quad - \sum_{i=na+2}^{nb} g(i/n)(f(i/n) - f((i-1)/n)) \end{aligned} \tag{10}$$

As $n \to \infty$, this last formula (10) approaches (8).

To verify (9), note that, in the first sum, b_i appears in a term with a_i and a_{i-1}, as in the final sum. The final sum thus matches the initial sum except in its initial and final terms. Those discrepancies are covered by the first two terms on the right.

(b) Let $B \equiv \{B(t) : t \geq 0\}$ be standard Browmian motion and let f be a real-valued function having a continuous derivative f' on $[a, b]$. Suppose that we define the stochastic integral

$$\int_a^b f(x)\, dB(x) \tag{11}$$

by an appropriate limit of the approximating sums

$$\sum_{i=na+1}^{nb} f(i/n)(B((i+1)/n - B(i/n)) \tag{12}$$

as $n \to \infty$, just as in (10). Use (9) and the continuity of the paths of Brownian motion to justify the formula

$$\int_a^b f(x)\, dB(x) = f(b)B(b) - f(a)B(a) - \int_a^b B(x)f'(x)\, dx \tag{13}$$

Given (12), we can apply (9) in the form of (10), where $g(i/n) = B(i/n)$. By continuity, the right side approaches (13).

(c) Find the full distribution of the discounted present value of a Brownian income stream, with discount rate r, i.e., of

$$D(r) \equiv \int_0^\infty e^{-rt}\, dB(t) \tag{14}$$

Hint: Apply Problem 1 on homework 9.

We can first apply the integration by parts formula above to get

$$D(r) = \int_0^\infty re^{-rt}B(t)\, dt \tag{15}$$

To be careful, we could first do the analysis on a finite interval $[0, b]$ and then let $b \to \infty$, but there are no difficulties, because $B(t)/t \to 0$ with probability one as $t \to \infty$, while the discounting term decays exponentially fast. As a generalization of a sum of normal random variables, the integral of a deterministic function with respect to Brownian motion is normally distributed. It thus suffices to calculate the mean and variance.

We can do that in two ways. First, directly in terms of stochastic integrals, we have mean 0 and

$$Var(D(r)) = Var\left(\int_0^\infty e^{-rt}\, dB(t)\right) = \int_0^\infty (e^{-rt})^2\, dt = \int_0^\infty e^{-2rt}\, dt = \frac{1}{2r}; \tag{16}$$

the analysis on page 645 of Ross, Exercise 10.25, or by formula (19) in the stochastic calculus notes (with $M = B$ and $\langle M\rangle(t) = t$, so that $d\langle M\rangle(t) = dt$).

The second way is by direct computation, as in Homework exercise 9.16. Let

$$Z(t) = re^{-rt}B(t) \ .$$

Then, exploiting $E[B(s)B(u)] = \min\{u, s\}$, we get

$$\begin{aligned}
E[D(r)^2] &= E\left(\int_0^\infty Z_s\,ds\right)^2 = E\left(\int_0^\infty Z_u\,du\right)\left(\int_0^\infty Z_s\,ds\right) \\
&= E\left(\int_0^\infty \int_0^\infty Z_u Z_s\,du\,ds\right) \\
&= E\left(\int_0^\infty \int_0^s Z_u Z_s\,du\,ds\right) + E\left(\int_0^\infty \int_s^\infty Z_u Z_s\,du\,ds\right) \\
&= \int_0^\infty \int_0^s ur^2 e^{-r(u+s)}\,du\,ds + \int_0^\infty \int_s^\infty sr^2 e^{-r(u+s)}\,du\,ds \\
&= \int_0^\infty e^{-rs}((1 - e^{-rs} - rse^{-rs})\,ds + \int_0^\infty rse^{-2rs}\,ds \\
&= \frac{1}{r} - \frac{1}{2r} - \frac{1}{4r} + \frac{1}{4r} = \frac{1}{2r}
\end{aligned}$$

Hence, $D(r) \stackrel{\mathrm{d}}{=} N(0, 1/2r)$.

Closing Remark. For full stochastic integrals, involving the integration of one stochastic process with respect to Brownian motion or some other martingale, the classical integration-by-parts formula has a correction term, e.g., see p. 155 of Karatzas and Shreve (1988). If we are integrating one continuous martingale X with respect to another Y, then the formula can be written as

$$\int_0^t X_s\,dY_s + X_tY_t - X_0Y_0 - \int_0^t Y_s\,dX_s - \langle X, Y\rangle_t \ ,$$

where the extra correction term $\langle X, Y\rangle_t$ is the quadratic variation, defined by

$$\langle X, Y\rangle_t \equiv \lim \sum_{i=1}^{n} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

using subscripts for time arguments and letting the limit as the partition of the time interval $[0, t]$ gets finer and finer.