

# IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

## SOLUTIONS to Homework Assignment 1: due, July 12, 2007

**Probability Review:** Read Chapters 1 and 2 in the textbook, *Introduction to Probability Models*, ninth edition, by Sheldon Ross. Please do the six problems marked with an asterisk and turn them in on Thursday. These problems are written out in case you do not yet have the textbook (available in the Columbia Bookstore). Extra problems are provided in case you need extra review. Solutions will be provided for all the following problems.

### CHAPTER 1

#### \*Problem 1.3.

A coin is to be tossed until a head appears twice in a row. What is the sample space for this experiment? If the coin is fair, what is the probability that it will be tossed exactly four times?

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The sample space is a set with countably infinitely many elements. The sample space is the set

$$S \equiv \{(e_1, e_2, \dots, e_n) : n \geq 2\},$$

where  $e_i \in \{H, T\}$  (i.e.,  $e_i$  is either heads or tails) and, in addition,  $e_n = e_{n-1} = H$  and, for  $1 \leq i \leq n-2$  (for  $n \geq 3$ ), if  $e_i = H$ , then necessarily  $e_{i+1} = T$ . Here are initial elements of  $S$ :  $\{H, H\}$ ,  $\{T, H, H\}$ ,  $\{T, T, H, H\}$ ,  $\{H, T, H, H\}$ ,  $\{T, T, T, H, H\}$ ,  $\{T, H, T, H, H\}$ ,  $\{H, T, T, H, H\}$ ,  $\{T, T, T, T, H, H\}$ , ...

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#### \*Problem 1.21

Suppose that 5% of men and 0.25% of women are color-blind. A color-blind person is chosen at random. What is the probability of this person being male? (Assume that there are an equal number of males and females.)

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Let  $C$  be the event that a person chosen at random is color-blind. Let  $M$  be the event that a random person is male. Let  $M^c$  be the event that a random person is female, the complement of the event  $M$ . Then

$$\begin{aligned} P(M|C) &= \frac{P(MC)}{P(C)} = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} \\ &= \frac{0.05 \times 0.5}{(0.05 \times 0.5) + (0.0025 \times 0.5)} = \frac{20}{21}. \end{aligned}$$

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#### Problem 1.22

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The calculation becomes easy if we view it in a good way. One might observe that the experiment has the form of tennis once deuce has been reached or table tennis (ping pong) after the score of 20 – 20 has been reached, so this is a situation that you may well have encountered previously.

Here is a good way to view it: Let successive trials consist of successive pairs of points. That is, we consider what happens after 2 plays, after 4 plays, and so on. We do that, because we observe that the game can only end after an even number of plays. Moreover, with this definition of a “trial,” the sequence of games becomes a sequence of Bernoulli trials (a sequence of independent and identically distributed (IID) Bernoulli random variables; see Section 2.2.1. The probability that each player wins one point on any trial is  $q = 2p(1 - p)$ . A total of  $2n$  points are played if the first  $(n - 1)$  trials all results in each player winning one point, and the  $n^{\text{th}}$  trial results in one of the players winning two points. We apply the independence to calculate the desired results.

The probability that  $2n$  points are needed is thus  $(2p(1 - p))^{n-1}(p^2 + (1 - p)^2)$ ,  $n \geq 1$ .

The probability that  $A$  eventually wins is

$$\begin{aligned} P(A \text{ wins}) &= \left( \sum_{n=1}^{\infty} (2p(1 - p))^{n-1} \right) \times p^2 \\ &= \frac{p^2}{1 - 2p(1 - p)}, \end{aligned}$$

using the well known property of geometric series:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}$$

for  $0 < x < 1$ .

#### Problem 1.31

Two dice are tossed, one green and one red. What is the conditional probability that the number on the green die is 6, given that the sum on the two dice is 7?

Extra part: What is the conditional probability that the number on the green die is 6, given that the sum on the two dice is 10?

Let  $S$  be the event that the sum is 7; let  $G$  be the event that the number on the green die is 6. Then

$$P(G|S) = \frac{P(GS)}{P(S)} = \frac{P(S|G)P(G)}{P(S)} = \frac{(1/6) \times (1/6)}{6/36} = \frac{1}{6}.$$

Now for the extra part. Let  $S$  now be the event that the sum is 10. Then

$$P(G|S) = \frac{P(GS)}{P(S)} = \frac{P(S|G)P(G)}{P(S)} = \frac{(1/6) \times (1/6)}{3/36} = \frac{1}{3}.$$

\*Problem 1.43

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Here is another application of Bayes' Theorem, again like the examples in Section 1.6. Let  $i$  be the event that coin  $i$  is selected. As usual, let  $H$  and  $T$  denote the events head and tail. We are told that  $P(H|i) = i/10$ ; we want to calculate the conditional probability the other way. We get

$$\begin{aligned}
 P(i = 5|H) &= \frac{P(H \cap 5)}{P(H)} \\
 &= \frac{P(H|5)P(5)}{\sum_{i=1}^{10} P(H|i)P(i)} \\
 &= \frac{5/100}{\sum_{i=1}^{10} i/100} \\
 &= \frac{5}{\sum_{i=1}^{10} i} \\
 &= \frac{5}{55} = \frac{1}{11} = 0.090909\dots
 \end{aligned}$$


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## CHAPTER 2

### Problem 2.9

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The probability mass function (pmf) gives the jumps in the cumulative distribution function (cdf). Let  $p(b)$  be the pmf. Then

$$p(b) = \begin{cases} 1/2 & b = 0 \\ 1/10 & b = 1 \\ 1/5 & b = 2 \\ 1/10 & b = 3 \\ 1/10 & b = 3.5 \end{cases}$$


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### Problem 2.20

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This is a multinomial distribution. Let  $T$  be the number of ordinary television sets purchased; let  $C$  be the number of color television sets purchased; and let  $B$  be the number of customers browsing. Then

$$P(T = 2, C = 2, B = 1 | T + C + B = 5) = \frac{5!}{2!2!1!} (0.2)^2 (0.3)^2 (0.5)^1 = 0.054$$

This could be approached by using the binomial distribution twice. First we consider the purchasers of ordinary TV's and all others. Then we divide up the others into purchasers of color TV's and browsers. That leads to the product of two binomial distributions, as follows:

$$P(T = 2, C = 2, B = 1 | T + C + B = 5) = P(T = 2, T^c = 3 | T + C + B = 5) P(C = 2, B = 1 | C + B = 3),$$

yielding the same result after calculation.

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\*Problem 2.32

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This problem is about approximating the binomial distribution by the Poisson distribution. Directly, the problem can be solved by the binomial distribution, but the binomial distribution with large  $n$  and small  $p$ , with  $np = \lambda$ , is approximately Poisson with mean  $\lambda$ . Let  $N$  be the number of prizes won. Then, first exactly and then approximately,

$$\begin{aligned} P(N = k) &= b(k; n, p) = b(k, 50, 0.01) = \frac{50!}{k!(50-k)!} (0.01)^k (0.99)^{50-k} \\ &\approx \text{Poisson}(k; \lambda = np) = \text{Poisson}(k; 0.5) = \frac{e^{-0.5} (0.5)^k}{k!} \end{aligned}$$

so that

$$\begin{aligned} P(N \geq 1) &= 1 - P(N = 0) \approx 1 - e^{-0.5} = 0.394 \\ P(N = 1) &= P(N = 1) \approx e^{-0.5} (0.5)^1 = 0.303 \\ P(N \geq 2) &= 1 - P(N = 0) - P(N = 1) \approx 0.394 - 0.303 = 0.091 \end{aligned}$$

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Problem 2.33

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We need the total probability to be 1. Hence we must solve

$$c \int_{-1}^1 (1 - x^2) dx = 1 .$$

Elementary calculus yields

$$c \left( x - \frac{x^3}{3} \right) \Big|_{-1}^1 = 1$$

or  $c(4/3) = 1$ , so that

$$c = 3/4 .$$

The cdf  $F$  is the integral of the pdf  $f$ , as on the bottom of page 34, so that

$$F(y) = \int_{-\infty}^y f(x) dx = \frac{3}{4} \int_{-1}^y (1 - x^2) dx = \frac{3}{4} \left( y - \frac{y^3}{3} + \frac{2}{3} \right)$$

for  $-1 \leq y \leq 1$ . Clearly,  $F(y) = 0$  for  $y < -1$  and  $F(y) = 1$  for  $y > 1$ .

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\*Problem 2.34

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To be a probability density function, its integral must be 1. So we must have

$$c \int_0^2 (4x - 2x^2) dx = c(2x^2 - 2x^3/3) = 1 ,$$

which implies that  $c = 3/8$ . Then

$$P(1/2 < X < 3/2) = \frac{3}{8} \int_{1/2}^{3/2} (4x - 2x^2) dx = \frac{11}{16}.$$

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Problem 2.39

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$$EX = 1 \times p(1) + 2 \times p(2) + 24 \times p(24) = \frac{1}{2} + \frac{2}{3} + \frac{24}{6} = \frac{31}{6} = 5\frac{1}{6}$$

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\*Problem 2.43

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This problem gives practice in the use of indicator random variables, the  $X_i$  here.

(a)

$$X = \sum_{i=1}^n X_i$$

(b)

$$EX_i = P(X_i = 1) = P(\text{ball } i \text{ chosen before all black balls}) = \frac{1}{m+1},$$

so

$$EX = \frac{n}{m+1}.$$

The key theoretical fact here is this: The expected value of a sum of random variables is the sum of the expectations, even if the random variables are dependent. There is such dependence in this example. See pages 49-51 in the textbook for additional discussion. In contrast, the variance of a sum of random variables is the sum of the variances only under the condition that the component random variables are independent.

The reason why  $P(\text{ball } i \text{ before all black balls}) = \frac{1}{m+1}$  is the following. Let  $Y$  be the first ball that is either ball  $i$  or a black ball. We then condition on the color of ball  $Y$ :

$$\begin{aligned} P(\text{ball } i \text{ before all black balls}) &= \underbrace{P(\text{ball } i \text{ before all black balls} \mid Y \text{ is red})}_{=1} P(Y \text{ is red}) \\ &\quad + \underbrace{P(\text{ball } i \text{ before all black balls} \mid Y \text{ is black})}_{=0} P(Y \text{ is black}) \\ &= P(Y \text{ is red}). \end{aligned}$$

This probability is easy to calculate: just before we draw ball  $Y$ , all black balls and ball  $i$  are still in the urn. So the probability that ball  $Y$  is ball  $i$  is  $1/(m+1)$ .

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Problem 2.48

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$$E[X^2] = \int_0^1 x^2 dx = \frac{1}{3}.$$

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Problem 2.50

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Use definition of the variance.

(i)

$$\text{Var}(cX) = E[(cX - E[cX])^2] = E[c^2(X - EX)^2] = c^2E[(X - EX)^2] = c^2\text{Var}(X)$$

(ii)

$$\text{Var}(c + X) = E[(c + X - E[c + X])^2] = E[(X - EX)^2] = \text{Var}(X).$$