IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

SOLUTIONS to Homework Assignment 5

The Exponential Distribution and the Poisson Process

Due on Wednesday, July 25 before class (10:00am); discussed at the recitation on Tuesday evening, July 24, 7:30-9:00pm in 633 Mudd.

Do the following exercises at the end of Chapter 5:

1. Since T is exponential with mean 1/2 hour,

$$P(T > t) = e^{-2t}, \quad t > 0 \ .$$

Hence,

(a)

$$P(T > 1/2) = e^{-2(1/2)} = e^{-1} \approx \frac{1}{2.71828} \ .$$

(b) By the lack-of-memory property (see Section 5.2.2),

$$P(T > 12.5|T > 12) = P(T > 1/2) = e^{-2(1/2)} = e^{-1} \ .$$

2. The time you spend in the system, say T, is the sum of 6 IID (independent and identically distributed) exponential random variables, each with mean $1/\mu$. Hence,

$$ET = 6/\mu \ .$$

3. The conditional distribution of X given that $X > 1$ is the same as the unconditional distribution of $1 + X$. Hence

$$E[X^2|X > 1] = E[(X + 1)^2] \ ;$$

i.e., (a) is correct.

4. (a) 0, (b) one way: $(1/3^3 = 1/27$, (c) 1/4 because

$$\begin{aligned}(A \text{ last to leave}) &= P(B \text{ served before } A) \times P(C \text{ served before } A|B \text{ served before } A) \\ &= (1/2) \times (1/2) = 1/4 \ .\end{aligned}$$

5. Let T be the lifetime of the radio. Then

$$P(T > t \text{ years}) = e^{-0.1t}, \quad t > 0 \ .$$

Hence, by the lack-of-memory property (again see Section 5.2.2),

$$P(T > 20 | T > 10) = P(T > 10) = e^{-0.1(10)} = e^{-1} \ .$$

8. Note that

$$\begin{aligned} E[1/r(X)] &= \int (1/r(x)) f(x) \, dx \\ &= \int \frac{1 - F(x)}{f(x)} f(x) \, dx \\ &= \int 1 - F(x) \, dx \\ &= E[X] \ . \end{aligned}$$

The last step is a very useful relation, as indicated in the hint; see page 580 of Ross.

9. The probability that machine 1 fails before time t is $1 - e^{-\lambda_1 t}$. The probability that machine 1 is still working at time t is $e^{-\lambda_1 t}$. Conditional on machine 1 working at time t, the probability that machine 1 fails first is $\lambda_1/(\lambda_1 + \lambda_2)$. Hence the final answer is

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t} \frac{\lambda_1}{(\lambda_1 + \lambda_2)} \ .$$

11. Use two facts: (1) the lack of memory property and (2) the fact that the minimum of independent exponential random variables is again exponential with a rate equal to the sum of the individual rates (rate equals reciprocal of the mean). Hence,

$$P(A_1) = P(X > min\{Y_1, \ldots, Y_n\}) = \frac{n\mu}{\lambda + n\mu} \ .$$

Moreover,

$$P(A_j | A_1 A_2 \ldots A_{j-1}) = \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu} \ .$$

Hence,

$$\begin{aligned} p &\equiv P(X > max\{Y_i\}) \\ &= P(A_1) P(A_2 | A_1) \ldots P(A_n | A_1 A_2 \ldots A_{n-1}) \\ &= \prod_{j=1}^{j=n} \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu} \ . \end{aligned}$$

When $n = 2$,

$$\begin{aligned} p &\equiv P(X > max\{Y_i\}) \\ &= \int_0^\infty P(max\{Y_i\} < X | X = x) \lambda e^{-\lambda x} \, dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty P(max\{Y_i\} < x)\lambda e^{-\lambda x}\, dx \\
&= \int_0^\infty (1-e^{-\mu x})^2 \lambda e^{-\lambda x}\, dx \\
&= \int_0^\infty (1-2e^{-\mu x}+e^{-2\mu x})\lambda e^{-\lambda x}\, dx \\
&= 1 - \frac{2\lambda}{\lambda+\mu} + \frac{\lambda}{\lambda+2\mu} \\
&= \frac{2\mu^2}{(\lambda+\mu)(\lambda+2\mu)}\ .
\end{aligned}
$$

26. (a) This is just the probability one exponential variable is less than another:

$$\frac{\mu_1}{\mu_1+\mu_3}$$

(b) You now have two independent events that must both happen, so it is the product of the probabilities:

$$\frac{\mu_1}{\mu_1+\mu_3} \times \frac{\mu_2}{\mu_2+\mu_3}$$

(c) The expected time at the first two servers is easy, but the expected time at the last server is tricky, because the other customer could be there still. So the expected total time is the sum of the three expected service times plus the expected service time of the initial customer at server 3 times the probability that he is still there when the arrival gets there; i.e.,

$$E[Time] = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} + \frac{\mu_1}{\mu_1+\mu_3}\frac{\mu_2}{\mu_2+\mu_3}(\frac{1}{\mu_3})\ .$$

(d) Let X_i be the service time at server i for customer X (new customer), $i = 1, 2, 3$. Let Y_j be the service time at server j for customer Y (old customer), $j = 2, 3$. We know that $X_i^{'}s$ and $Y_j^{'}s$ are independent and exponentially distributed (but with different means). Let T be the total time required. Let T_1= time until next event happens, and let $T^{'} = T - T_1$. Then $ET = ET_1 + ET^{'}$. Clearly $T_1 = min(X_1, Y_2)$, so $ET_1 = \frac{1}{\mu_1+\mu_2}$. We remark that the distribution of T_1 is independent of the event $\{X_1 < Y_2\}$; see property 5 of concise summary.) Now we condition to find the mean of T':

$$
\begin{aligned}
ET^{'} &= E[T^{'}|X_1 > Y_2]\cdot P(X_1 > Y_2) + E[T^{'}|X_1 < Y_2]\cdot P(X_1 < Y_2) \\
&= E[T^{'}|X_1 > Y_2]\cdot \frac{\mu_2}{\mu_1+\mu_2} + E[T'|X_1 < Y_2]\cdot \frac{\mu_1}{\mu_1+\mu_2}
\end{aligned}
$$

Note that $E[T^{'}|X_1 > Y_2]$ is already computed in part (c), the only unknown here is $E[T^{'}|X_1 < Y_2]$. Note that

$$E[T^{'}|X_1 < Y_2] = \frac{1}{\mu_2} + ET_2 + ET^{"}$$

Where T_2 = the time until next event happens after customer Y finishes his service at server 2, then $T^{"}$ = the time from the next event happens until customer X leaves the system. Since $T_2 = min(X_2, Y_2)$, we have $ET_2 = \frac{1}{\mu_2+\mu_3}$. And also:

$$
\begin{aligned}
E[T^{"}] &= E[T^{"}|X_2 > Y_3]\cdot P(X_2 > Y_3) + E[T^{"}|X_2 < Y_3]\cdot P(X_2 < Y_3) \\
&= \frac{1}{\mu_3}\cdot\frac{\mu_3}{\mu_2+\mu_3} + \frac{2}{\mu_3}\cdot\frac{\mu_2}{\mu_2+\mu_3}
\end{aligned}
$$

It's not necessary to simplify this result.

39. Recall Problem 4 (e) on the midterm exam. We can use a normal approximation by virtue of the central limit theorem. See Section 2.7 if you need a refresher. By the assumptions, the lifetime is the sum of 196 IID exponential random variables each with mean 1/2.5 years, i.e.,

$$S_{196} = X_1 + \cdots + X_{196} ,$$

where $EX_i = 1/2.5$. Thus,

(a) the mean is

$$E[S_{196}] = 196/2.5 = 78.4 .$$

(b) and the variance is

$$Var[S_{196}] = 196(1/2.5)^2 = 31.36 .$$

For parts (c) - (e), use the normal approximation, which follows from the central limit theorem; see Section 2.7. Use the normal probabilities in Table 2.3. Let Z be a standard normal random variable (with mean 0 and variance 1).

(c)

$$P(S_{196} < 67.2) \approx P(Z < (67.2 - 78.4)/5.6) = P(Z < -2.0) = 0.0227 .$$

(d)

$$P(S_{196} > 90) \approx P(Z > (90 - 78.4)/5.6) = P(Z > 2.07) = 0.0192 .$$

(e)

$$P(S_{196} > 100) \approx P(Z > (100 - 78.4)/5.6) = P(Z > 3.857) = 0.00006 .$$

42.
(a) This is just the sum of four IID exponential random variables:

$$E[S_4] = \frac{4}{\lambda} .$$

(b) The conditioning event says that there are two events up to time 1. Hence,

$$E[S_4|N(1) = 2] = 1 + E[\text{time for} \quad 2 \quad \text{more events}] = 1 + \frac{2}{\lambda} .$$

(c) Recall that a Poisson process has independent increments. So,

$$E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)] = E[N(2) - N(0)] = E[N(2)] = 2\lambda .$$