IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

SOLUTIONS to Homework Assignment 7

Due on Monday, August 6; discussed in Recitation on Sunday, August 5.

Chapter 7: Renewal Theory and its Applications

In Ross, read Sections 7.1-7.3 up to, but not including, Example 7.8. Skip Remark (ii) in Section 7.2 and Examples 7.1 and 7.3. Also Read Section 7.4 up to, but not including, Example 7.14. (The total required reading is approximately 13 pages.)

Do the following exercises at the end of Chapter 7. You need not turn in problems with answers in the back.

1. Hint: See the beginning of Section 7.2.

The defining relation is (7.1). The property that does hold is (7.2). This example illustrates that correctness depends on whether or not the inequalities are strict or not.

The answers are (a) yes, (b) no, and (c) no.

It is easy to see that (a) is equivalent to (7.2), while the others are not. It is not difficult to give concrete examples.

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2. Hint: See Sections 2.2.4 and 7.2. Recall that the sum of independent Poisson random variables has a Poisson distribution.

Since X_n is Poisson with mean μ, S_n is Poisson with mean $n\mu$. From (7.3),

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= \sum_{k=0}^{k=\lfloor t \rfloor} \frac{e^{-n\mu}(n\mu)^k}{k!} - \sum_{k=0}^{k=\lfloor t \rfloor} \frac{e^{-(n+1)\mu}((n+1)\mu)^k}{k!}, \end{aligned}$$

where $\lfloor t \rfloor$ is the greatest integer less than or equal to t.

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3. Hint: See Example 7.2 and following Remark (i).

The answer is in the back of the book.

The one-to-one relationship is easy to see when we use Laplace transforms. For a nonnegative random variable X, its Laplace transform is $E[e^{-sX}]$, where s is a new variable (in general regarded as a complex variable, but that is not crucial here). If X has probability density function (pdf) f, then the Laplace transform of X coincides with the Laplace transform of the pdf f, denoted by $\hat{f}(s)$; i.e.,

$$\hat{f}(s) = E[e^{-sX}] = \int_0^\infty e^{-sx} f(x)\, dx \ .$$

It is known that there is a one-to-one correspondence betweeen a pdf and its Laplace transform. Moreover, it turns out that the Laplace transform of the cdf F, where

$$F(t) = \int_0^t f(u)\, du, \quad t \ge 0 \ ,$$

is $\hat{f}(s)/s$. In addition, if S_n is the sum of n independent and identically distributed (IID) random variables, each with pdf f, then

$$E[e^{-sS_n}] = E[e^{-sX_1}]^n = \hat{f}(s)^n.$$

Since the renewal function $m(t)$ satisfies (see Section 7.2)

$$m(t) = \sum_{n=1}^{\infty} P(S_n \le t) \ ,$$

the Laplace transform of $m(t)$, defined by

$$\hat{m}(s) = \int_0^\infty e^{-st} m(t)\, dt \ ,$$

satisfies the relation

$$\hat{m}(s) = \sum_{n=1}^{\infty} \hat{f}(s)^n/s = \frac{\hat{f}(s)}{s(1-\hat{f}(s))} \ .$$

We need the s in the denominator because we have the cdf of S_n, not its density, when we look at $P(S_n \le t)$; i.e., the Laplace transform of $P(S_n \le t)$ is $\hat{f}(s)^n/s$.

From the last equation, we see that we can solve for $\hat{f}(s)$ given $\hat{m}(s)$ by

$$\hat{f}(s) = \frac{s\hat{m}(s)}{1 + s\hat{m}(s)} \ .$$

Hence we do indeed have the one-to-one reflationship between the pdf of X_n, denoted by f, and the renewal function m, as claimed.

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4. Hint: Consider the special case of deterministic times between renewals.

In general the answers to (a)-(c) are all NO. However, the answers are all YES for the special case of a Poisson process. It turns out that the answers are NEVER yes for renewal processes if both processes are not Poisson, but that is somewhat hard to prove.

To construct a specific example, let one process be a Poisson process with rate 1, and let the other renewal process have constant times between renewals, with value 1 for all n.

(a) No. If the first interarrival time in $N(t)$ is $1/4$, then the second is sure to be less than $3/4$. Hence the interarrival times in $N(t)$ are not independent.

(b) No. The probability that the first interarrival time is 1 is e^{-1}. The probability that the second interarrival time is 1 is necessarily different. The second interarrival time can be exactly 1 only if no Poisson arrivals have occurred by time 2. That probability is e^{-2}.

(c) No, because of parts (a) and (b).

It is important that all three of these properties do hold when the two component processes N_1 and N_2 are independent Poisson processes. Then the superposition process N is itself a Poisson process (a very special case of a renewal process).

7.

The mean time between successive jobs is $3 + 2 = 5$ months. The rate at which Mr. Smith gets new jobs is 1 per every 5 months or 2.4 jobs per year.

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8. Hint: Look at Section 7.3.

The answer is in the book.

21.

Note that this example is an $M/G/1/0$ queue, where the service times are IID with a general distribution. In this model there is a single server, but no extra waiting space. There is either one customer in the system or zero.

Let m be the mean of the general service time. The proportion of time that the server is busy is

$$\frac{m}{m + (1/\lambda)} \ .$$

We obtain this by using the renewal-reward-process framework (see the middle of page 418). A cycle is a service time plus the following interarrival time. We want to compute the expected reward per cycle divided by the expected length of a cycle. Reward is earned at rate 1 when the server is busy. The expected reward per cycle is the expected service time. Thus we get the formula above.

26. Hint: Look at Section 7.4.

This is a renewal-reward process problem. The long-run average cost is the expected cost per cycle divided by the expected length of a cycle. A cycle is the interval between successive arrivals of the train. Since customer arrivals occur according to a Poisson process, successive cycles are IID.

The expected length of the cycle is $\frac{N}{\lambda} + K$. The interarrival time between successive arrivals is exponential with mean $1/\lambda$. Thus the time until N arrivals occurs has mean N/λ.

The expected cost per cycle is c times the sum of the numbers of customers times the expected time per cycle with that number of customers. Until the N^{th} customer arrives, there are i customers present for an expected duration of $1/\lambda$. After the N^{th} customer arrives the expected number of customers in the system is $N + \lambda t$ at time t, $0 \leq t \leq K$. The integral from 0 to K is $NK + \lambda K^2/2$. Thus the expected cost per cycle is

$$\begin{aligned} E[\text{cost per cycle}] &= c[\frac{0}{\lambda} + \frac{1}{\lambda} + \cdots \frac{N-1}{\lambda} + NK + \lambda K^2/2] \\ &= c[\frac{(N-1)N}{2\lambda} + NK + \lambda K^2/2] \ . \end{aligned}$$

Hence the long-run expected cost is

$$\frac{c[\frac{(N-1)N}{2\lambda} + NK + \lambda K^2/2]}{\frac{N}{\lambda} + K} \ .$$

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38. Hint: Look at Section 7.4.

This again is a renewal-reward process problem.

(a) The proportion of his driving time spent driving from A to B is

$$\frac{E[T_{A,B}]}{E[T_{A,B}] + E[T_{B,A}]} \ ,$$

where $E[T_{A,B}]$ is the expected time to drive from A to B, while $E[T_{B,A}]$ is the expected time to drive from B to A.

To find $E[T_{A,B}]$ and $E[T_{B,A}]$, we use the elementary formula $d = rt$ (distance = rate $\times$ time). Let S be the driver's random speed driving from A to B. Then

$$\begin{aligned} E[T_{A,B}] &= \frac{1}{20}\int_{40}^{60} E[T_{A,B}|S = s]\, ds \\ &= \frac{1}{20}\int_{40}^{60} \frac{d}{s}\, ds \\ &= \frac{d}{20}(ln(60) - ln(40)) \\ &= \frac{d}{20}(ln(3/2) \ . \end{aligned}$$

Similarly,

$$\begin{aligned} E[T_{B,A}] &= \frac{1}{2}E[T_{B,A}|S = 40] + \frac{1}{2}E[T_{B,A}|S = 60] \\ &= \frac{1}{2}(\frac{d}{40} + \frac{d}{60}) \\ &= \frac{d}{48} \end{aligned}$$

(b) Assume that a reward is earned at rate 1 per unit time whenever he is driving at a rate of 40 miles per hour, we can again apply the renewal reward approach. If p is the long-run proportion of time he is driving 40 miles per hour,

$$p = \frac{(1/2)d/40}{E[T_{A,B}] + E[T_{B,A}]} = \frac{1/80}{\frac{1}{20}ln(3/2) + 1/48} \ .$$
