IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

SOLUTIONS to Homework Assignment 8

Arbitrage, due on Monday, August 13

Read Sections 10.4.1 and 10.4.2 in Ross.

Do the following exercises at the end of Chapter 10.

1. Exercise 10.12. corrections: We must add the strike price for the option: Suppose that the option gives you the opportunity, but not the obligation, to purchase a share of stock at time 1 for $K = 125$ dollars per share again in present value dollars). That was left out of the question by mistake. You have the opportunity to either buy or sell any amount of the stock or the option at time 0. As stated, the option costs C dollars per share.

answers: (a) First suppose that you purchase x shares of stock and y shares of the option, allowing these to be positive (buying) or negative (selling). Let C be the cost per share of the option. Let us find values of x and y so that the outcome is the same if the stock goes up or down. If the stock goes up, we gain $100x + (25 - C)y$. If the stock goes down, then we gain $-25x - Cy$. We set these equal, so we get the equation

$$100x + 25y - Cy = -25x - Cy \ ,$$

which reduces to

$$25y = -125x \quad \text{or} \quad y = -5x \ .$$

It follows that our gain is $-25x + 5xC = 5x(C - 5)$. In order to have no arbitrage (sure gain), we must have $C = 5$.

(b) If C = 4, then our gain is $-5x = y$, no matter what happens. Hence we should buy $-x$ shares of the stock (i.e., sell x shares of the stock) and buy $y = -5(-x) = 5x$ shares of the option in order to have a gain of $5x$. We gain an amount proportional to $|x|$ regardless of the outcome. We naturally would let x be as large as possible.

(c) If C = 10, then our gain is $25x = -5y$, no matter what happens. Since $C > 5$, now we buy the stock and sell the options. We should buy x shares of the stock and buy $y = -5x$ shares (i.e., sell 5x shares) of the option in order to have a gain of $25x$. We gain an amount proportional to $|x|$ regardless of the outcome.

(d) We find probabilities so that the expected change in the stock value is 0. Since we gain 100 if the stock goes up and lose 25 if the stock goes down, the probability of going up must be $p = 1/5$, while the probability of going down must be 4/5. To check, note that the expected gain from purchasing one share of the stock is

$$(1/5)100 + (4/5)(-25) = 20 - 20 = 0 \ .$$

Hence the cost C of the option per share must be such that the expected gain is 0 also with these probabilities, but we have

$$(1/5)(25 - C) + (4/5)(-C) = 0$$

or $C = 5$, just as we found above.

(e) **extra problem**: We are now asked how the analysis for parts (a) and (d) would change if the interest rate were given as $r = 0.25$ per year, with the time period being one year. Moreover, assume that the returns given at the end of the year are expressed as dollars at time 1. Remember that we buy the stock and the option at time 0, but if we elect to exercise the option, then we make that stock purchase at time 1.

(a') As before, suppose that you purchase x shares of stock and y shares of the option, allowing these to be positive (buying) or negative (selling). Let C be the cost per share of the option. Let us find values of x and y so that the outcome is the same if the stock goes up or down. If the stock goes up, we gain $[150/(1+r)-50]x+[(150-125)/(1+r)-C)y$. That equals $[150x + 25y]/(1 + r) - 50x - Cy$ If the stock goes down, then we gain $[25/(1 + r) - 50]x - Cy$. We set these equal, so we get

$$y = -5x \ ,$$

just as before, independent of r. It follows that our gain now is $(25x)/(1 + r) - 50x + 5Cx = 5x(C - 10 + 5/(1 + r))$. In order to have no arbitrage (sure gain), we must have

$$C = 10 - \frac{5}{1+r} = 10 - \frac{5}{1.25} = 10 - 4 = 6$$

(d') In present value (at time 0), the stock values at time 1 are $150/(1+r) = 150/1.25 = 120$ and $25/(1 + r) = 25/1.25 = 20$. So the expected return is 0 if $p120 + (1 - p)20 - 50 = 0$ or $100p = 30$, so that we should have $p = 0.3$ and $1 - p = 0.7$. For the option, we have $p25/1.25 + (1 - p)0 - C = 20p - 6$, which equals 0 if and only if $p = 0.3$.

2. Exercise 10.13.

If the outcome is i, then we win $x_i o_i$ from our bet on i, but we lose $\sum_{j:j\neq i} x_j$ from our other bets. Hence our total winnings are

$$\begin{aligned} x_i o_i - \sum_{j:j\neq i} x_j &= \frac{o_i(1+o_i)^{-1} - \sum_{j:j\neq i}(1+o_j)^{-1}}{1 - \sum_j (1+o_j)^{-1}} \\ &= \frac{(1+o_i)(1+o_i)^{-1} - \sum_j (1+o_j)^{-1}}{1 - \sum_j (1+o_j)^{-1}} \\ &= \frac{1 - \sum_j (1+o_j)^{-1}}{1 - \sum_j (1+o_j)^{-1}} = 1 \end{aligned}$$

Thus we win 1 no matter what happens. Hence if we bet cx_i on i for all i, then we win c. We can win arbitrarily much without risk.

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3. Exercise 10.14. Unlike in problem 10.12 above, we are given the strike price of the option. With the option, we have the opportunity, but not the obligation, to buy the stock at time 1 for the present value, which is 100. As in problem 10.12, assume that the returns in period 1 are expressed in present value dollars, so that we do not separately need to consider the interest rate.

Introduce probabilities for each of the three outcomes. Let p_1 be the probability that the new price is 50; let p_2 be the probability that the new price is 100; then $1 - p_1 - p_2$ is the probability that the new price is 200. Purchasing the stock will be a fair bet if

$$100 = 50p_1 + 100p_2 + 200(1 - p_1 - p_2)$$

or, equivalently, if

$$3p_1 + 2p_2 = 2$$

From this alone, we see that we must have $p_1 \leq 2/3$.

(a) Now, turning to the option, with a strike price of $K = 120$, we see that it is a fair bet if

$$(1 - p_1 - p_2)80 = c$$

Let $p_2 = (2 - 3p_1)/2$. Putting this in the last equation gives

$$(1 - p_1 - 1 + (3/2)p_1)80 = c$$

or

$$p_1 = c/40 \quad \text{and} \quad p_2 = (80 - 3c)/80 \quad \text{and} \quad 1 - p_1 - p_2 = c/80 \ .$$

We get legitimate probabilities for all three provided that $3c \leq 80$. By the Arbitrage Theorem on p. 635, we have no arbitrage provided that $c \leq 80/3$. If $c > 80/3$, then there is an arbitrage opportunity.

(b) Now, turning to the option with a strike price of $K = 80$, we see that it is a fair bet if

$$(1 - p_1 - p_2)120 + 20p_2 = c$$

Let $p_2 = (2-3p_1)/2$, just as before (by considering the stock). Putting this in the last equation gives

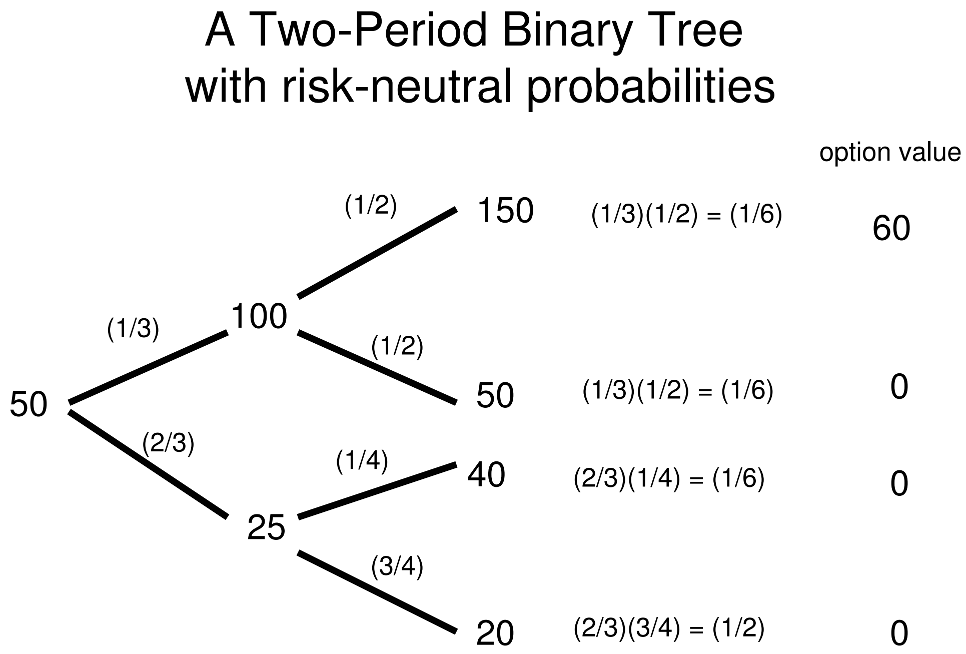
$$p_1 = (c - 20)/30 \quad \text{and} \quad p_2 = (40 - c)/20 \quad \text{and} \quad 1 - p_1 - p_2 = (c - 20)/60$$

These will be between 0 and 1 if and only if $20 \leq c \leq 40$.

4. A Two-Period Binary Tree

(a) We can apply the Arbitrage Theorem - Theorem 10.1 - on page 635 to conclude that it suffices to find risk-neutral prices for each of the three random events in the tree. These probabilities are shown in parentheses on the arcs of the tree in Figure below. The three random events are then understood to be independent Bernoulli experiments.

We use the independence to calculate the probabilities of each of the four possible outcomes at stage 2. We multiply the probabilities on the connecting arcs to get the final probabilities for each outcome. We can then calculate the expected value of the option. Since the call option gives us the opportunity to purchase the stock at time 2 for 90 dollars per share, we will only elect to do so if the stock price goes up twice in a row, i.e., in the case in which the stock price is 150 at time 2. We realize a profit of 60 dollars per share in that case, but we get 0 in all other cases. The value of the option - price at stage 2 minus strike price K if positive, and otherwise 0 - is given at the right in Figure . The expected value of the stock option using the



risk neutral probabilities is $60 \times (1/6) = 10$ dollars per share. So the fair (arbitrage-free) price of the option is 10 dollars per share.

(b) We now develop a hedging strategy. We can do this in more than one way. We give two methods. The second method employs method in the Lecture Notes on the Binomial Lattice Model.

Method 1

Let X_i denote the stock price at the end of period i. Consider 4 possible actions:

- A_0 Put one dollar in the bank and end up with one dollar in all possible scenarios
- A_1 Buy one share of stock at time 0 and sell it at time 1
- A_2 Buy one share of the stock at time 1 and sell at time 2, if the stock goes up to 100 at time 1
- A_3 Buy one share of the stock at time 1 and sell at time 2, if the stock goes down to 25 at time 1

Let z_i be the amount (number of shares) done of action i. We want to find values of z_i that replicate the option under every possible event. To do so, we construct the following table: Table 1.

X_1	X_2	A_0	A_1	A_2	A_3	option
100	150	1	50	50	0	60
100	50	1	50	-50	0	0
25	40	1	-25	0	15	0
25	20	1	-25	0	-5	0

Table 1: Creation of the hedging strategy to replicate the option.

Considering both times 1 and 2, we see that there are four possible outcomes. We need to replicate the option in each of these cases. That leads to four equations in four unknowns. each equation is of the form $A_0 z_0 + A_1 z_1 + A_2 z_2 + A_3 z_3 = v$. Here are the four equations:

$$\begin{aligned} z_0 + 50z_1 + 50z_2 + 0z_3 &= 60 \\ z_0 + 50z_1 - 50z_2 + 0z_3 &= 0 \\ z_0 - 25z_1 + 0z_2 + 15z_3 &= 0 \\ z_0 - 25z_1 + 0z_2 - 5z_3 &= 0 \end{aligned} \quad (1)$$

By subtracting the first two equations, we get $100z_2 = 60$ or $z_2 = 3/5$. By subtracting the last two equations, we get $20z_3 = 0$ or $z_3 = 0$. It makes sense to have $z_3 = 0$ because the option is unaffected by the random event at time 1 after the stock price has dropped. If we substitute into the first and third equations, then we get two equations in two unknowns:

$$\begin{aligned} z_0 + 50z_1 &= 30 \\ z_0 - 25z_1 &= 0 \end{aligned}$$

Subtracting these, we get $75z_1 = 30$ or $z_1 = 30/75 = 6/15 = 2/5$ and $z_0 = 10$. The variable z_0 is the price of the option. We get $z_0 = 10$, which is consistent with the expected value under the risk-neutral probabilities. We replicate the option itself with $(z_0, z_1, z_2, z_3) = (10, 2/5, 3/5, 0)$.

Method 2.

Being more systematic, we also can apply the single-period two-alternative analysis in Section 3.2 of the Binomial Lattice Model Lecture Notes, starting at the end and working back. We find a portfolio (α, β) at each decision point, where α is the number of shares of the stock and β is the amount of the risk-free asset. We use formulas (12) and (13) in those lecture notes. First, after one period, when the stock price is 100 after going up, we get $(\alpha, \beta) = (3/5, -30)$; the option is worth 30 at this stage. Second, after one period, when the stock price is 25 after going down, we get $(\alpha, \beta) = (0, 0)$; the option is worth 0 at this stage. Third, at the very beginning, when the stock price is 50, we have another two-alternative tree with option values 30 for up and 0 for down. At this point, we get $(\alpha, \beta) = (2/5, -10)$; the option at the beginning is worth 10. We have assumed that the interest rate here is $r = 1$. It is easy to include positive interest rate.

5. The Put-Call Option Parity Formula

It is convenient to write the formula as

$$S + P - C = Ke^{-rt} ,$$

because it equates the present values of two payments. We initially buy the stock, but we always sell it at the end of the time period. For the left side, we buy a share of the stock at price S, we buy the put option for P and sell the call option for C. The left side is what we pay. For that payment we own the stock. Now observe that exactly one of the options is exercised at the end of the period in all cases, both of which cause us to sell the stock. If the stock price at the end of the period is greater than K, then the call option we sold initially will be exercised and we will have to sell the stock to meet the sold call option. On the other hand, if the stock price is less than K at the end of the period, then the call option will not be exercised, but we will elect to exercise the put option. We will then sell the stock. At the end of the period we will have received K. So we sell the stock and receive payment K at the end of the period in both cases. The present value of that payment at the end is Ke^{-rt}. So the initial payment $S + P - C$ should equal the present value of the final payment to us, which is Ke^{-rt}. That gives the equation above.