IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

Homework Assignment 9: Brownian motion

Due on Monday, August 13

In Ross, read Sections 10.1-10.3 and 10.6. (The total required reading there is approximately 11 pages.) Also reread pages 72-73 and read the seven pages on *multivariate normal distributions* in Wikipedia (handed out).

This is a long assignment. You only need turn in the nine asterisked problems below. However, early problems help do later problems.

I. The probability law of a stochastic process.

The probability law of a stochastic process is usually specified by giving all the finite-dimensional distributions (f.d.d.'s). Let $\{X(t) : t \geq 0\}$ be a stochastic process, i.e., a collection of random variables indexed by the parameter t, which is usually thought of as time. Then the f.d.d.'s of this stochastic process are the collection of k-dimensional probability distributions of the random vectors $(X(t_1), \ldots, X(t_k))$, over all possible positive integers k and all possible vectors $(t_1, \ldots, t_k)$ and $(x_1, \ldots, x_k)$ with $0 < t_1 < \cdots < t_k$.

The multivariate cdf of $X(t_1), \ldots, X(t_k)$ is given by

$$F_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k) \equiv P(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_k) \leq x_k) ,$$

where $0 < t_1 < \cdots < t_k$ and $(x_1, \ldots, x_k)$ is an element of $\mathbb{R}^k$.

We sometimes go further and define a stochastic process as a random function, where time is the argument of the function. The probability law then is the probability measure on the space of functions, but that needs to be made precise. In considerable generality, a consistent set of finite-dimensional distributions will determine a unique probability measure on the space of functions. A bit more can be seen if you Google *stochastic process* or *Kolmogorov extension theorem.*

II. multivariate normal distributions.

Brownian motion is a Gaussian process, which is a stochastic process whose finite-dimensional distributions are multivariate normal distributions. So it is good to consider the multivariate normal distribution.

1. linear combinations.

Show that linear combinations of multivariate normal random variables have a multivariate normal distribution. That is, suppose that $(X_1, \ldots, X_m)$ has a multivariate normal distribution, and consider $Y_i = \sum_{j=1}^{m} A_{i,j} X_j + B_i$ for $i = 1, \ldots, k$, where $A_{i,j}$ and B_i are constants (non-random real numbers). Show that $(Y_1, \ldots, Y_k)$ has a multivariate distribution as well, and characterize that distribution. In matrix notation, let $Y = AX + B$, where Y and B are $k \times 1$, A is $k \times m$ and X is $m \times 1$. (We want X and Y to be column vectors. Formally, we would write $X = (X_1, \ldots, X_m)^T$ and $Y = (Y_1, \ldots, Y_k)^T$, where T denotes transpose. For a matrix A, $(A^T)_{i,j} = A_{j,i}$.)

This demonstration can be done in **two ways**: (1) exploiting the definition in terms of independent normal random variables on the bottom of page 72 and (2) using the joint moment generating function, as defined on pages 72-73. Hint: See the Wikidedia account.

2. marginal distributions.

Suppose that $1 \leq k < m$. Show that $(X_1, \ldots, X_k)$ has a multivariate normal distribution if $(X_1, \ldots, X_m)$ has a multivariate normal distribution. Hint: Apply the previous exercise.

3. marginal distributions.

Show that (X_1, X_2) need not have a multivariate normal distribution if X_1 and X_2 each are normally distributed real-valued random variables. Hint: look at page 2 of the Wikipedia article.

4*. covariance and independence.

Show that real-valued random variables X and Y are uncorrelated if they are independent, but they may be dependent if they are uncorrelated. Hint: Look at page 52. For the last part, give an example. Hint: Consider a probability distribution attaching probability one to finitely many points in the plane.

5. covariance and independence.

Show that random variables X and Y are independent if (X, Y) has a bivariate normal distribution and $Cov(X, Y) = 0$. Hint: Again look at page 52. Also look at the bivariate normal pdf in the Wikipedia notes.

6*. higher moments.

Use the moment generating function of the standard normal distribution to determine the third and fourth moments of $N(0, \sigma^2)$. Hint: see page 67.

7. conditional distributions.

Suppose that (X, Y) have a $k+m$-dimensional normal distribution, where $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_m)$. What is the conditional distribution of X given that $Y = a = (a_1, \ldots, a_m)$? Write the pdf of this conditional distribution, assuming that the covariance matrix Σ_a is nonsingular. Hint: See the Wikipedia account. This can be derived by looking at the multivariate pdf, and observing that, when we condition, the joint pdf becomes a new pdf of the same form when we fix some of the variables. Except for constant terms, the exponent becomes a new quadratic form in a subset of the variables. It is possible to solve for the new normal pdf by the technique of completing the square.

8*. more conditional distributions.

Suppose that (X_1, X_2) have a 2-dimensional normal distribution, where $E[X_i] = \mu_i$, $Var(X_i) = \sigma_i^2$ and $Cov(X_1, X-2) = \sigma_{1,2}$. What are $E[X_1|X_2 = a]$ and $Var[X_1|X_2 = a]$?. Hint: This is a special case of the previous problem.

Do the following exercises at the end of Chapter 2.

9*. Exercise 2.76

10*. Exercise 2.77

11*. Exercise 2.78

III. Brownian Motion

12. finite-dimensional distributions.

(a) Show that the f.d.d.'s of Brownian motion $\{B(t) : t \geq 0\}$ are multivariate normal by applying Problem 1 above.

(b) Directly construct the probability density function of $B(t_1), \ldots, B(t_k)$ for Brownian motion $\{B(t) : t \geq 0\}$.

Do the following exercises at the end of Chapter 10. You need not turn in the exercises with answers in the back.

Exercise 10.1. (answer in back)

13. Exercise 10.2*. Hint: Apply the result in the middle of page 604.

Exercise 10.3. (answer in back)

14. Exercise 10.6*. Hint: See Section 10.2.

15. Exercise 10.7*. Hint: See Section 10.2, just as for the problem above.

two optional extra problems

16. Let

$$Y(t) \equiv \int_0^t B(s)\, ds \ ,$$

where $B \equiv \{B(t) : t \geq 0\}$ is standard Brownian motion. Find:

(a) $E[Y(t)]$

(b) $E[Y(t)^2]$

(c) $E[Y(s)Y(t)]$ for $0 < s < t$.

17. Find the conditional distribution of $Y(t)$ given $B(t) = x$, where these processes are as in the previous problem. Hint: Thinking of the integral as the limit of sums, we can apply problem 1 to find the general form of the joint distribution of $(Y(t), B(t))$. That leaves only the computation of the means, variances and covariances. We can then apply Problem 8 above. That still leaves some calculations to do.