IEOR 4701: Stochastic Models in Financial Engineering

Summer 2007, Professor Whitt

SOLUTIONS to Homework Assignment 9: Brownian motion

In Ross, read Sections 10.1-10.3 and 10.6. (The total required reading there is approximately 11 pages.) Also reread pages 72-73 and read the seven pages on *multivariate normal distributions* in Wikipedia.

This is a long assignment. You only need turn in the nine asterisked problems below. However, early problems help do later problems.

I. The probability law of a stochastic process.

The probability law of a stochastic process is usually specified by giving all the finite-dimensional distributions (f.d.d.'s). Let $\{X(t) : t \geq 0\}$ be a stochastic process; i.e., a collection of random variables indexed by the parameter t, which is usually thought of as time. Then the f.d.d.'s of this stochastic process are the collection of k-dimensional probability distributions of the random vectors $(X(t_1), \ldots, X(t_k))$, over all possible positive integers k and all possible vectors $(t_1, \ldots, t_k)$ and $(x_1, \ldots, x_k)$ with $0 < t_1 < \cdots < t_k$.

The multivariate cdf of $X(t_1), \ldots, X(t_k)$ is given by

$$F_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k) \equiv P(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_k) \leq x_k) \; ,$$

where $0 < t_1 < \cdots < t_k$ and $(x_1, \ldots, x_k)$ is an element of $\mathbb{R}^k$.

We sometimes go further and define a stochastic process as a random function, where time is the argument of the function. The probability law then is the probability measure on the space of functions, but that needs to be made precise. In considerable generality, a consistent set of finite-dimensional distributions will determine a unique probability measure on the space of functions. A bit more can be seen if you Google *stochastic process* or *Kolmogorov extension theorem.*

II. multivariate normal distributions.

Brownian motion is a Gaussian process, which is a stochastic process whose finite-dimensional distributions are multivariate normal distributions. So it is good to consider the multivariate normal distribution.

1. linear combinations.

Show that linear combinations of multivariate normal random variables have a multivariate normal distribution. That is, suppose that $(X_1, \ldots, X_m)$ has a multivariate normal distribution, and consider $Y_i = \sum_{j=1}^{m} A_{i,j} X_j + B_i$ for $i = 1, \ldots, k$, where $A_{i,j}$ and B_i are constants (non-random real numbers). Show that $(Y_1, \ldots, Y_k)$ has a multivariate distribution as well, and characterize that distribution. In matrix notation, let $Y = AX + B$, where Y and B are $k \times 1$, A is $k \times m$ and X is $m \times 1$. (We want X and Y to be column vectors. Formally, we would write $X = (X_1, \ldots, X_m)^T$ and $Y = (Y_1, \ldots, Y_k)^T$, where T denotes transpose. For a matrix A, $(A^T)_{i,j} = A_{j,i}$.)

This demonstration can be done in **two ways**: (1) exploiting the definition in terms of independent normal random variables on the bottom of page 72 and (2) using the joint moment generating function, as defined on pages 72-73. Hint: See the Wikidedia account.

First, if we say that $X \equiv (X_1, \ldots, X_m)^T$ is multivariate normal if $X = CZ + D$, where $Z \equiv (Z_1, \ldots, Z_n)^T$ is a vector of *independent standard normal random variables*, for some n. Here X and D are $m \times 1$, while Z is $n \times 1$ and C is $m \times n$. As a consequence, X has mean vector $\mu_X = D$ and covariance matrix

$$\Sigma_X = E[(CZ)(CZ)^T] = E[(CZ)Z^T C^T] = CE[ZZ^T]C^T = CIC^T = CC^T \ .$$

Given that Y is a linear combination of X, we can write $Y = AX + B$ in matrix notation, where Y and B are $k \times 1$, while X is $m \times 1$ and C is $k \times m$. Then we can write $Y = ACZ + (AD + B)$, which shows that Y can be written as a linear function of Z, just like X, using the matrix multiplier AC and the additive constant vector $AD + B$. That means that Y has mean vector $\mu_Y = AD + B$ and covariance matrix

$$\Sigma_Y = E[(Y - \mu_Y)(Y - \mu_Y)^T] = E[A(X - \mu_X)(X - \mu_X)^T A^T] = A\Sigma_X A^T \ .$$

Alternatively, using the moment generating function (mgf), we can write

$$\phi_Y(t) \equiv \phi_{Y_1,\ldots,Y_k}(t_1, \ldots, t_k) \equiv E[e^{t_1 Y_1 + \cdots + t_k Y_k}] \ ,$$

but now we substitute in for Y_i using $Y_i = \sum_{j=1}^m A_{i,j} X_j + B_i$ to get

$$\phi_{Y_1,\ldots,Y_k}(t_1, \ldots, t_k) = \exp\{\sum_{i=1}^k t_i B_i\} \phi_{X_1,\ldots,X_m}(\sum_{i=1}^k t_i A_{i,1}, \ldots, \sum_{i=1}^k t_i A_{i,m}) \ ,$$

where

$$\phi_{X_1,\ldots,X_m}(s_1, \ldots, s_m) = \exp\{\sum_{j=1}^m s_j \mu_j + \frac{1}{2}\sum_{j=1}^m \sum_{k=1}^m s_j s_k Cov(X_j, X_k)\} \ .$$

Now we need to substitute into the exponent and rearrange terms. We need to the essentially the same linear algebra in the exponent. When we do so, again we see that the mgf has the form of the multivariate normal distribution, but with altered parameters. That leads to a new derivation of what we did with the direct matrix operations above.

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2. marginal distributions.

Suppose that $1 \leq k < m$. Show that $(X_1, \ldots, X_k)$ has a multivariate normal distribution if $(X_1, \ldots, X_m)$ has a multivariate normal distribution.

We can apply the previous problem, because the truncation to the first k variables can be written as a linear function. We can write $Y \equiv (X_1, \ldots, X_k)$. Then Y = AX for $X = (X_1, \ldots, X_m)$, where A is $k \times m$ with $A_{i,i} = 1$ for $0 \leq i \leq k$ and $A_{i,j} = 0$ in all other cases.

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3. marginal distributions.

Show that (X_1, X_2) need not have a multivariate normal distribution if X_1 and X_2 each are normally distributed real-valued random variables. Hint: look at page 2 of the Wikipedia article.

Following Wikipedia, let $Y = X$ if $|X| > 1$ and let $Y = -X$ if $|X| \leq 1$. If X and Y are normally distributed, then the random vector (X, Y) does **not** have a bivariate normal distribution. This constructed bivariate density has support on the union of the two sets $(x, y) : x = y > 1$ and $(x, y) : x = -y \quad \text{and} \quad |x| \leq 1$. Bivariate normal densities either are positive over the entire plane or are degenerate, concentrating on some line. Here there is degeneracy, but it does not fall on a line.

4*. covariance and independence.

Show that real-valued random variables X and Y are uncorrelated if they are independent, but they may be dependent if they are uncorrelated. For the last part, give an example. Hint: Consider a probability distribution attaching probability one to finitely many points in the plane.

As stated on page 52, independence of X and Y implies that (and is actually equivalent to) $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$ for all real-valued functions f and g for which the expectations are well defined. As a special case, we get $E[XY] = E[X]E[Y]$, which is equivalent to $Cov(X, Y) = 0$. To show that uncorrelated does **not** imply independence, we can construct a counterexample. Let (X, Y) take the values $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$, $(0, 0)$, each with probability $1/5$. Then X and Y are uncorrelated, but they are dependent. For example, $P(Y = 0|X = 0) = 1 \neq 1/5 = P(Y = 0)$.

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5. covariance and independence.

Show that random variables X and Y are independent if (X, Y) has a bivariate normal distribution and $Cov(X, Y) = 0$. Hint: Again look at page 52. Also look at the bivariate normal pdf in the Wikipedia notes.

It is easy to see that the bivariate probability density function and the two-dimensional mgf factor into the product of two functions when the covariance is 0. Each of these factorizations implies independence. First, consider the pdf. From the bivariate pdf displayed in the Wikipedia article, we see that it factors if the covariance is 0, because the exponential of a sum is the product of the exponentials. The two separate exponentials contain the functions associated with X and Y, respectively That implies independence, as stated on page 52. Alternatively, we can establish independence using the mgf. In general, we have

$$\phi_{X,Y}(t_1, t_2) = \exp t_1\mu_1 + t_2\mu_2 + \frac{\sigma_1^2}{2}t_1^2 + \frac{\sigma_2^2}{2}t_2^2 + Cov(X, Y)t_1t_2 \ .$$

However, if the covariance term is 0, then we have

$$\phi_{X,Y}(t_1, t_2) = \phi_X(t_1)\phi_Y(t_2) \ .$$

That turns out to imply independence of X and Y.

6*. higher moments.

Use the moment generating function of the standard normal distribution to determine the third and fourth moments of $N(0, \sigma^2)$. Hint: see page 67.

By symmetry, $E[N(0, \sigma^2)^3] = 0$. By differentiating the mgf, using the chain rule, we confirm this. We get $E[N(0, \sigma^2)^4] = 3\sigma^4$ by the same reasoning.

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7. conditional distributions.

Suppose that (X, Y) have a $k+m$-dimensional normal distribution, where $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_m)$. What is the conditional distribution of X given that $Y = a = (a_1, \ldots, a_m)$? Write the pdf of this conditional distribution, assuming that the covariance matrix Σ_a is nonsingular. Hint: See the Wikipedia account. This can be derived by looking at the multivariate pdf, and observing that, when we condition, the joint pdf becomes a new pdf of the same form when we fix some of the variables. Except for constant terms, the exponent becomes a new quadratic form in a subset of the variables. It is possible to solve for the new normal pdf by the technique of completing the square.

The main fact is that this conditional distribution is multivariate normal. It thus suffices to exhibit the conditional mean, say $m(a) \equiv (E[X_1|Y = a], \ldots, E[X_k|Y = a])$ and the associated $k \times k$ covariance matrix Σ_a. These are displayed in the Wikipedia account. Let μ_1 be the mean vector of X; let μ_2 be the mean vector of Y; Let $\Sigma_{1,1}$ be the covariance matrix of X; let $\Sigma_{2,2}$ be the covariance matrix of Y; and let $\Sigma_{1,2}$ be the matrix of covariance between variables in X and Y; i.e., $\Sigma_{1,2}$ is the $k \times m$ matrix with $(i,j)^{th}$ entry $Cov(X_i, Y_j)$. Then the k-dimensional mean vector is

$$\mu_a = \mu_1 + \Sigma_{1,2}\Sigma_{2,2}^{-1}(a - \mu_2) \ ,$$

while the $k \times k$ covariance matrix is

$$\Sigma_a = \Sigma_{1,1} - \Sigma_{1,2}\Sigma_{2,2}^{-1}\Sigma_{2,1} \ ,$$

where $\Sigma_{2,1}$ is the transpose of $\Sigma_{1,2}$. The pdf is displayed in the Wikipedia; just use the mean vector μ_a and covariance matrix Σ_a above.

8*. more conditional distributions.

Suppose that (X_1, X_2) have a 2-dimensional normal distribution, where $E[X_i] = \mu_i$, $Var(X_i) = \sigma_i^2$ and $Cov(X_1, X-2) = \sigma_{1,2}$. What are $E[X_1|X_2 = a]$ and $Var[X_1|X_2 = a]$? Hint: This is a special case of the previous problem.

We just apply the formulas above in this lower-dimensional case. We get the conditional mean

$$m(a) = \mu_1 + \sigma_{1,2}\sigma_{2,2}^{-1}(a - \mu 2) \ ,$$

while the 1×1 covariance matrix is just the variance, i.e.,

$$\sigma_a^2 = \sigma_{1,1}^2 - \sigma_{1,2}(\sigma_{2,2}^2)^{-1}\sigma_{2,1} \ ,$$

where $\sigma_{2,1} = \sigma_{1,2} = Cov(X_1, X_2)$ and the matrix inverse $(\sigma_{2,2}^2)^{-1}$ reduces to the simple reciprocal.

Do the following exercises at the end of Chapter 2.

9*. Exercise 2.76

$$\begin{aligned} E[XY] &= E[X]E[Y] = \mu_x \mu_y \\ E[X^2Y^2] &= E[X^2]E[Y^2] = (\mu_x^2 + \sigma_x^2)(\mu_y^2 + \sigma_y^2) \\ Var[XY] &= E[X^2Y^2] - (E[X]E[Y])^2 = (\mu_x^2 + \sigma_x^2)(\mu_y^2 + \sigma_y^2) - (\mu_x \mu_y)^2 \\ &= \mu_x^2 \mu_y^2 + \mu_x^2 \sigma_y^2 + \sigma_x^2 \mu_y^2 + \sigma_x^2 \sigma_y^2 - (\mu_x \mu_y)^2 \\ &= \mu_x^2 \sigma_y^2 + \sigma_x^2 \mu_y^2 + \sigma_x^2 \sigma_y^2 \end{aligned}$$

as claimed.

10*. Exercise 2.77

Note that (U, V) is a linear function of (X, Y), where $U = X + Y$ and $V = X - Y$, so that (U, V) is necessarily normally distributed, by Problem 1 above. Hence, by Problem 5 above. it suffices to show that U and V are uncorrelated. The required calculation is:

$$E[UV] = E[(X + Y)(X - Y)] = E[X^2 - Y^2] = (\mu^2 + \sigma^2) - (\mu^2 + \sigma^2) = 0 \ ,$$

so that U and V are uncorrelated, and thus independent.

11*. Exercise 2.78

(a) Look at $\phi(t_1, \ldots, t_n)$ after setting $t_j = 0$ for all j not equal to i. That is,

$$\phi_{X_i}(t_i) = \phi(t_1, \ldots, t_n) \quad \text{for} \quad t_j = 0 \quad \text{if} \quad j \neq i \ .$$

To see that this must be the right thing to do, recall the representation

$$\phi(t_1, \ldots, t_n) \equiv E\left[\exp\left\{t_1 X_1 + \cdots + t_n X_n\right\}\right] \ .$$

(b) We use the fact that a joint mgf determines a multivariate probability distribution uniquely. This critical fact is stated, but not proved, on page 72. On the other hand, if the random variables are independent, then the righthand side holds, by virtue of Proposition 2.3 on page 52. Hence the righthand side is the joint mgf in the case that the one-dimensional random variables are independent. The random variables must then actually be independent, because there is no other joint distribution with that mgf.

III. Brownian Motion

12. finite-dimensional distributions.

(a) Show that the f.d.d.'s of Brownian motion $\{B(t) : t \geq 0\}$ are multivariate normal by applying Problem 1 above.

Use the fact that Brownian motion has independent normal increments. The increments $B(t_{i+1}) - B(t_i)$ are normally distributed with mean 0 and variance $t_{i+1} - t_i$. Each variable $B(t_i)$ is the sum of the previous increments:

$$B(t_i) = (B(t_1) - B(0)) + (B(t_2) - B(t_1)) + \cdots + (B(t_i) - B(t_{i-1})) \ .$$

Hence, the random vector $B(t_1), \ldots, B(t_k)$ is a linear function of the independent normal increments. So we can apply problem 1 above to deduce that the f.d.d.'s of Brownian motion are indeed multivariate normal.

(b) Directly construct the probability density function of $B(t_1), \ldots, B(t_k)$ for Brownian motion $\{B(t) : t \geq 0\}$.

This is done in the book in (10.3) on top of page 628.

Do the following exercises at the end of Chapter 10. You need not turn in the exercises with answers in the back.

Exercise 10.1. (answer in back)

This seems simple, but it is not quite as simple as it appears. First, $B(s) \stackrel{\mathrm{d}}{=} N(0, s)$, i.e., $B(s)$ is normally distributed with mean 0 and variance s. Similarly, $B(t) \stackrel{\mathrm{d}}{=} N(0, t)$. The difficulty is that these two random variables $B(s)$ and $B(t)$ are DEPENDENT. So we cannot compute the distribution of the sum by doing a convolution.

However, we can exploit the independent increments property to rewrite the sum as a sum of independent random variables. By adding and subtracting $B(s)$, we have $B(t) = B(s) + [B(t) - B(s)]$, where $B(t) - B(s)$ is independent of $B(s)$ by the independent increments property of Brownian motion. Hence

$$B(s) + B(t) = 2B(s) + [B(t) - B(s)] \ . \tag{1}$$

This representation is better because it is the sum of two independent random variables. We could compute its distribution by doing a convolution, but we will use another argument.

In whatever way we proceed, we will want to use the stationary increments property to deduce that

$$B(t) - B(s) \stackrel{\mathrm{d}}{=} B(t-s) - B(0) = B(t-s) \stackrel{\mathrm{d}}{=} N(0, t-s) \ .$$

We now invoke a general property about multivariate normal distributions. A linear function of normal random variables is again a normal random variable. (This is true even with dependence. This is true in all dimensions.) We thus know that $B(s) + B(t)$ is normal or, equivalently, $2B(s) + [B(t) - B(s)]$ is normal.

However, we do not need that general result, because we can represent (1) above, which tells us that we have the sum of two independent normal random variables. We know that is normally distributed by virtue of Example 2.46 on p. 70 (see problem 1 above). Either way, we know we have a normal distribution. A normal distribution is determined by its mean and variance. Since $E[B(t)] = 0$, it is elementary that

$$E[B(s) + B(t)] = 0 \quad \text{or} \quad E[2B(s) + [B(t) - B(s)]] = 0 \ .$$

Hence, finally, it suffices to compute the variance of $B(s)+B(t) = 2B(s)+[B(t)-B(s)]$. The second representation is easier, because the random variables are independent. For independent random variables, the variance of a sum is the sum of the variances. Hence

$$Var(2B(s)+B(t-s)) = Var(2B(s))+Var(B(t-s)) = 4Var(B(s))+Var(B(t-s)) = 4s+(t-s) = 3s+t \ .$$

Hence $B(s)+B(t) \stackrel{\text{d}}{=} N(0, 3s+t)$, i.e., is normally distributed with mean 0 and variance $3s+t$.

13. Exercise 10.2*.

This can be viewed as an application of problem 8 above. The conditional distribution $X(s) - A$ given that $X(t_1) = A$ and $X(t_2) = B$ is the same as the conditional distribution of $X(s - t_1)$ given that $X(0) = 0$ and $X(t_2 - t_1) = B - A$, which (by eq. 10.4) is normal with mean $\frac{s-t_1}{t_2-t_1}(B - A)$ and variance $\frac{s-t_1}{t_2-t_1}(t_2 - s)$. Hence the desired conditional distribution is normal with mean $A + \frac{s-t_1}{t_2-t_1}(B - A)$ and variance $\frac{s-t_1}{t_2-t_1}(t_2 - s)$

Exercise 10.3. (answer in back)

One approach is to use martingales: First, we can write the expectation as the expectation of a conditional expectation:

$$E[B(t_1)B(t_2)B(t_3)] = E[E[B(t_1)B(t_2)B(t_3)|B(r), 0 \le r \le t_2]] \ .$$

Now we evaluate the inner conditional expectation:

$$E[B(t_1)B(t_2)B(t_3)|B(r), 0 \le r \le t_2] = B(t_1)B(t_2)E[B(t_3)|B(t_2)] = B(t_1)B(t_2)^2 \ .$$

Hence the answer, so far, is

$$E[B(t_1)B(t_2)B(t_3)] = E[B(t_1)B(t_2)^2] \ .$$

We now proceed just as above by writing this expectation as the expectation of a conditional expectation:

$$E[B(t_1)B(t_2)B(t_3)] = E[B(t_1)B(t_2)^2] = E[E[B(t_1)B(t_2)^2|B(r), 0 \le r \le t_1]] \ .$$

We next evaluate the inner conditional expectation:

$$E[B(t_1)B(t_2)^2|B(r), 0 \le r \le t_1] = B(t_1)E[B(t_2)^2|B(t_1)] \ ,$$

where,

$$B(t_2)^2 = [B(t_1) + B(t_2) - B(t_1)]^2 = B(t_1)^2 + 2B(t_1)[B(t_2) - B(t_1)] + [B(t_2) - B(t_1)]^2 \ ,$$

so that

$$\begin{aligned}B(t_1)E[B(t_2)^2|B(t_1)] &= B(t_1)^3 + 2B(t_1)^2E[B(t_2) - B(t_1)|B(t_1] + B(t_1)E[B(t_2) - B(t_1)]^2 \\ &= B(t_1)^3 + 2B(t_1)^2 \times 0 + B(t_1)E[B(t_2 - t_1)]^2 \\ &= B(t_1)^3 + B(t_1)(t_2 - t_1)\end{aligned}$$

Now taking expected values again, we get

$$\begin{aligned}E[B(t_1)B(t_2)B(t_3)] &= E[B(t_1)B(t_2)^2] = E[E[B(t_1)B(t_2)^2|B(r), 0 \le r \le t_1]] \\ &= E[B(t_1)^3] + E[B(t_1)(t_2 - t_1)] = 0 + 0 \ ,\end{aligned}$$

using the fact that $E[B(t_1)^3] = 0$, because the third moment of a normal random variable with mean 0 is necessarily 0, because of the symmetry. Hence, $E[B(t_1)B(t_2)B(t_3)] = 0$.

Another longer, but less complicated (because it does not use conditional expectations), argument is to break up the expression into independent pieces: Replace $B(t_3)$ by $B(t_2) + [B(t_3) - B(t_2)]$ and then by $B(t_1) + [B(t_2) - B(t_1)] + [B(t_3) - B(t_2)]$. And replace $B(t_2)$ by $B(t_1)+[B(t_2)-B(t_1)]$. Then substitute into the original product and evaluate the expectation:

$$E[B(t_1)B(t_2)B(t_3)] = E[B(t_1)(B(t_1)+[B(t_2)-B(t_1)])(B(t_1)+[B(t_2)-B(t_1)]+[B(t_3)-B(t_2)])] \ .$$

Or, equivalently,

$$\begin{aligned}E[B(t_1)B(t_2)B(t_3)] &= E[x_1(x_1 + x_2)(x_1 + x_2 + x_3)] \\ &= E[x_1^3 + 2x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3] \ ,\end{aligned}$$

where $x_1 \equiv B(t_1)$, $x_2 = B(t_2) - B(t_1)$ and $x_3 = B(t_3) - B(t_2)$. Since the random variables x_1, x_2 and x_3 are independent (the point of the construction),

$$E[x_1^3 + 2x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3] = E[x_1^3] = E[B(t_1)^3] = 0 \ ,$$

because the third moment of a normal random variable with mean 0 is 0.

14. Exercise 10.6*.

The probability of recovering your purchase price is the probability that a Brownian Motion goes up c by time t. We thus want to exploit the distribution of the maximum, as discussed on page 630. Hence the desired probability is

$$1 - P\{\max_{0 \le s \le t} X(s) \ge c\} = 1 - \frac{2}{\sqrt{2\pi}} \int_{\frac{c}{\sqrt{t}}}^{\infty} e^{-y^2/2} dy \ ,$$

see the display in the middle of page 630. Recall that X(t) is distributed as $N(0,t)$, which in turn is distributed as $\sqrt{t}N(0,1)$. So we can recast this in terms of the tail of a standard normal random variable, and get something we could look up in the table.

15. Exercise 10.7*.

We can find this probability by conditioning on $X(t_1)$

$$P\{\max_{t_1 \le s \le t_2} X(s) > x\} = \int_{-\infty}^{\infty} P\{\max_{t_1 \le s \le t_2} X(s) > x | X(t_1) = y\} \frac{1}{\sqrt{2\pi t_1}} e^{y^2/2t_1} dy \ (*)$$

Where

$$P\{\max_{t_1 \le s \le t_2} X(s) > x | X(t_1) = y\} = P\{\max_{0 \le s \le t_2 - t_1} X(s) > x - y\} \text{ if } y < x$$

$$= 1 \text{ if } y > x$$

Substitution of the above equation into $(*)$ now gives the required result when one uses the following,

$$P\{\max_{0 \le s \le t_2 - t_1} X(s) > x - y\} = 2P\{X(t_2 - t_1) > x - y\}$$

Where $X(t_2 - t_1) \sim N(0, t_2 - t_1)$

two optional extra problems

16. Let

$$Y(t) \equiv \int_0^t B(s)\, ds \ ,$$

where $B \equiv \{B(t) : t \ge 0\}$ is standard Brownian motion. Find:

(a) $E[Y(t)]$

(b) $E[Y(t)^2]$

(c) $E[Y(s)Y(t)]$ for $0 < s < t$.

In this problem, we want to take the expected value of integrals. In doing so, we want to conclude that the expectation can be moved inside the integrals; i.e., the expectation of an integral is the integral of the expected value. That makes sense intuitively because the integral is just the limit of sums, and the expected value of a sum of random variables is just the sum of the expectations. However, in general, what is intuitively obvious might not actually be correct. For the integrals, we use Fubini's theorem and Tonelli's theorem (from measure theory). But we will not dwell on these mathematical details; we will assume that we can take the expectation inside the integral. Although there are exceptions, this step is usually correct. This step tends to be a technical detail.

(a) As discussed above, we just take the expectation inside the integral, so this first part is easy. We get

$$EY_t = \int_0^t EB_s\, ds = 0 \ .$$

(b) To do this, we use a trick

$$\begin{aligned} EY_t^2 &= E\left(\int_0^t B_s\, ds\right)^2 = E\left(\int_0^t B_r\, dr\right)\left(\int_0^t B_s\, ds\right) \\ &= E\left(\int_0^t \int_0^t B_r B_s\, dr\, ds\right) \\ &= E\left(\int_0^t \int_0^s B_r B_s\, dr\, ds\right) + E\left(\int_0^t \int_s^t B_r B_s\, dr\, ds\right) = 2\int_0^t \int_0^s EB_r B_s\, dr\, ds \\ &= 2\int_0^t \int_0^s r\, dr\, ds = \int_0^t s^2\, ds = t^3/3 \ . \end{aligned}$$

(c) Clearly $EY_sY_t = EY_s^2 + EY_s(Y_t - Y_s)$, so we only have to compute the second term. To do this, we imitate the computation in (b)

$$EY_s(Y_t - Y_s) = E\left(\int_0^s B_r\, dr \cdot \int_s^t B_u\, du\right)$$

$$
\begin{aligned}
&= \int_0^s \int_s^t EB_r B_u \, du \, dr \\
&= \int_0^s \int_s^t r \, du \, dr = (t-s) \int_0^s r \, dr = (t-s)s^2/2
\end{aligned}
$$

17. Find the conditional distribution of $Y(t)$ given $B(t) = x$, where these processes are as in the previous problem. Hint: Thinking of the integral as the limit of sums, we can apply problem 1 to find the general form of the joint distribution of $(Y(t), B(t))$. That leaves only the computation of the means, variances and covariances. We can then apply Problem 8 above. That still eaves some calculations to do.

As indicated in the hint, a linear function of multivariate normal random variables is normal. The integral is a continuous linear function of normals, and so it too is normal. (It can be approached as a limit of sums.) Thus, the joint distribution of (Y_t, B_t) is bivariate normal. Since the conditional distribution of a multivariate normal given a marginal distribution is also normal, the conditional distribution will be normal. Hence it suffices to compute the conditional mean and variance. To compute the mean and variance, we note that, by Exercise 6.1,

$$
E(Y_t | B_t = z) = \int_0^t E(B_s | B_t = z) \, ds = \frac{z}{t} \int_0^t s \, ds = t \cdot \frac{z}{2}
$$

For the second moment, we again use the trick from the previous exercise, and the hint.

$$
\begin{aligned}
E(Y_t^2 | B_t = z) &= 2 \int_0^t \int_0^s E(B_r B_s | B_t = z) \, dr \, ds \\
&= 2 \int_0^t \int_0^s \frac{rz}{t} \frac{sz}{t} + \frac{r(t-s)}{t} \, dr \, ds
\end{aligned}
$$

$$
\begin{aligned}
E(B_r B_s \mid B_t = z) &= E(\ E(B_r B_s \mid B_s, B_t = z) \mid B_t = z) \\
&= E(\ B_s \, E(B_r \mid B_s, B_t = z) \mid B_t = z) \\
&= E(\ B_s \, E(B_r \mid B_s) \mid B_t = z) \\
&= E(B_s \frac{r}{s} B_s \mid B_t = z) \\
&= \frac{r}{s} E(B_s^2 \mid B_t = z) \\
&= \frac{r}{s} \{Var(B_s \mid B_t = z) + E^2(B_s \mid B_t = z)\} \\
&= \frac{r}{s} (\frac{s(t-s)}{t} + \frac{s^2 z^2}{t^2}) \\
&= \frac{rsz^2}{t^2} + \frac{r(t-s)}{t}
\end{aligned}
$$

The first term in the integral leads to the square of the mean of $(Y_t | B_t = z)$ so

$$
Var(Y_t | B_t = z) = \frac{2}{t} \int_0^t \frac{s^2}{2} (t-s) \, ds = \frac{1}{t} \left(\frac{t^3}{3} \cdot t - \frac{t^4}{4} \right) = t^3/12
$$