IEOR 6711: Stochastic Models, I

Fall 2008, Professor Whitt, Final Exam

There are three questions, each with several parts.

1. The BAD Lunch Deli (40 points)

Santiago Balsero, Andrew Ahn and Zuxuan Deng have joined forces to form their own delicatessen, the BAD (Balsero-Ahn-Deng) Lunch Deli, where they sell lunches. The BAD Lunch Deli mainly serves three items: sandwiches, salads and pan pizzas. These are provided at separate service stations. Santiago runs Service Station 1: the sandwich counter; Andrew runs Service Station 2: the salad counter; and Zuxuan runs Service Station 3: the pizza counter. Customers are served at each station one at a time in order of arrival. It takes a random time to serve each customer at each station. On average, it takes 1 minute to make a sandwich or a salad, while it takes 3 minutes to make a pizza.

During a typical lunch period, it has been observed that customers arrive at the BAD Lunch Deli at a rate of 30 per hour (1 every 2 minutes). The arriving customers go to each of the stations in approximately equal proportions. However, one our of four customers who get a sandwich afterwards go to the salad station, one out of three customers who get a salad afterwards go to the sandwich station, and one out of six customers who get a pizza go to the salad station. Everybody else leaves the BAD Lunch Deli service area after receiving service.

(a) Let $Q_i(t)$ be the number of people waiting or being served at station i. Assume that there is unlimited waiting space. Make additional assumptions (consistent with the description above) so that the stochastic process $\{(Q_1(t), Q_2(t), Q_3(t)) : t \geq 0\}$ becomes a CTMC. **Let these assumptions remain in force hereafter.**

(b) Prove or disprove: If $N_1(t)$ counts all arrivals to station 1 in the interval $[0, t]$, coming from outside or inside, then the stochastic process $\{N_1(t) : t \geq 0\}$ is a Poisson process under the assumptions of part (a).

(c) Prove or disprove: Under the assumptions of part (a), if the system starts empty, then the time until the first departure is distributed as the sum of two independent exponential random variables.

(d) What is the limiting probability distribution of $(Q_1(t), Q_2(t), Q_3(t))$ as $t \to \infty$ under the assumptions in part (a)?

(e) Which station is the bottleneck, i.e., which station is most congested in the long run? What is the expected number of customers there (waiting and being served) in steady state?

(f) Prove or disprove: The answer in part (d) remains unchanged if the interarrival times and service times have the specified means and retain the independence properties that follow from the assumptions of part (a), but the distributions of the interarrival times and service times are otherwise allowed to be general.

(g) Prove or disprove: If we let the initial random vector $(Q_1(0), Q_2(0), Q_3(0))$ have the limiting distribution in part (d), then the CTMC in part (a) becomes a reversible stochastic process (if we allow time to extend throughout the interval $(-\infty, +\infty)$).

(h) Prove that the answer in part (d) is correct. (Explain and give as much supporting detail. Best would be to state and prove the key supporting theorem.)

2. **Failure and Repair** (35 points)

A certain mechanism, which is intended to operate continuously, can fail in three different ways. If the mechanism is operational at time t, then a type-i failure will occur in the interval $(t, t+h)$ with probability $p_i(h)$, where $p_i(h) = \lambda_i h + o(h)$ as $h \downarrow 0$. (Failures can only occur when the mechanism is operational.) Upon failure, repair is performed, whose duration is an exponential random variable with mean depending on the failure type. The mean repair time for a type-i failure is $1/\mu_i$. When the repair is finished, the mechanism is operating again.

(a) Suppose that the mechanism is initially functioning. Let T_1 be the time of the first failure and let Z_1 be the type of the first failure. Prove or disprove: The random variables T_1 and Z_1 are independent.

(b) Find the long-run proportion of time that the mechanism is not working because of a type-2 failure.

(c) Let $X(t)$ be the state of the mechanism at time t, where the state indicates whether or not the mechanism is functioning and, if not, the type of failure being addressed. Indicate whether or not the stochastic process $\{X(t) : t \geq 0\}$ can be represented as each of the following kinds of stochastic processes:

(i) continuous-time Markov chain (CTMC)

(ii) birth-and-death (BD) process

(iii) reversible CTMC (assuming an appropriate inital distribution)

(iv) alternating renewal process

(v) semi-Markov process

(vi) regenerative process

(We are asking 6 different questions.) Briefly explain in each case.

(d) Construct a stochastic process $\{W(t) : t \geq 0\}$ with the same probability law as the stochastic process $\{X(t) : t \geq 0\}$ in part (c), such that the state changes of $\{W(t) : t \geq 0\}$ all occur at times that are a subsequence of a Poisson process.

(e) Prove that the probability law of the stochastic process $\{W(t) : t \geq 0\}$ constructed in part (d) is indeed the same as the probability law of the stochastic process $\{X(t) : t \geq 0\}$ in part (c).

(f) Give two alternative explicit expressions for the conditional probability that the system is operational at time t given that a type-2 failure was being repaired at time 0.

(g) Prove or disprove: The answer in part (b) remains unchanged if the type-i repair times are uniformly distributed over the interval $[0, 2/\mu_i$, $1 \leq i \leq 3$, instead of exponentially distributed, as stated.

3. **The City Soil Store** (25 points)

In order to supplement his income, an engineering professor has decided to open a soil store, where he sells soil to people in the City who want to grow their own gardens. Suppose that customers arrive at the soil store according to a renewal process having a continuous interarrival-time cumulative distribution function F with finite mean $1/\lambda$. Suppose that the amounts of soil purchased by successive customers come from a sequence of independent and identically distributed random variables $\{X_n : n \geq 1\}$ with continuous c.d.f. G having mean m. In addition, the random amounts purchased and the interarrival times are mutually independent.

Since there is a fixed cost for each delivery, in addition to a cost per quantity of soil, the professor has decided to use the classic (s, S) inventory policy: Whenever a purchase brings the inventory level below the level s, an order is placed to bring the inventory level up to the level S, where $S > s > 0$. Otherwise no order is placed. The orders to replenish the soil stock can be filled rapidly, so assume that the orders can be filled instantaneously. Moreover, assume that $P(X_n > s) = 0$, so that all orders can be fully satisfied.

(a) Give an expression for the expected value of each order quantity.

(b) State and prove a theorem that justifies your answer in part (a).

(c) Give an expression for the long-run proportion of time that the inventory level exceeds x for each $x \geq 0$.

(d) Let $I(t)$ be the inventory level at time t. Prove or disprove: The probability $P(I(t) > x)$ converges to the answer given in part (c) as $t \to \infty$.

(e) Indicate how the expression in part (c) can be efficiently computed.