

# IEOR 6711: Stochastic Models I

Fall 2012, Professor Whitt

## Solutions to Homework Assignment 11 on DTMC's

**Problem 4.40** Consider a segment of a sample path beginning and ending in state  $i$ , with no visit to  $i$  in between, i.e, the vector  $(i, j_1, j_2, j_3, \dots, j_{n-1}, j_n = i)$ , where  $j_k \neq i$  for the non-end states  $j_k$ . Going forward in time, the probability of this segment is

$$\pi_i P_{i,j_1} P_{j_1,j_2} P_{j_2,j_3} \cdots P_{j_{n-1},i}.$$

The probability, say  $p$ , of the reversed sequence  $(i, j_{n-1}, j_{n-2}, j_{n-3}, \dots, j_1, j_0 = i)$  under the reverse DTMC with transition matrix

$$\overleftarrow{P}_{i,j} \equiv \frac{\pi_j P_{j,i}}{\pi_i}$$

is

$$p = \pi_i \overleftarrow{P}_{i,j_{n-1}} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i}.$$

However, successively substituting in the reverse-chain transition probabilities, we get

$$\begin{aligned} p &= \pi_i \frac{\pi_{j_{n-1}} P_{j_{n-1},i}}{\pi_i} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i} \\ &= P_{j_{n-1},i} \pi_{j_{n-1}} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i} \\ &= P_{j_{n-1},j_{n-2}} \pi_{j_{n-1}} \frac{\pi_{j_{n-2}} P_{j_{n-2},j_{n-1}}}{\pi_{j_{n-1}}} \overleftarrow{P}_{j_{n-1},j_{n-2}} \overleftarrow{P}_{j_{n-2},j_{n-3}} \cdots \overleftarrow{P}_{j_1,i} \\ &= P_{j_{n-1},i} P_{j_{n-2},j_{n-1}} P_{j_{n-3},j_{n-2}} \cdots P_{j_1,j_2} \pi_{j_1} \overleftarrow{P}_{j_1,i} \\ &= P_{j_{n-1},i} P_{j_{n-2},j_{n-1}} P_{j_{n-3},j_{n-2}} \cdots P_{j_1,j_2} P_{i,j_1} \pi_i \\ &= \pi_i P_{i,j_1} P_{j_1,j_2} P_{j_2,j_3} \cdots P_{j_{n-1},i}. \end{aligned}$$

**Problem 4.41 (a)** The reverse time chain has transition matrix

$$\overleftarrow{P}_{i,j} \equiv \frac{\pi_j P_{j,i}}{\pi_i}$$

To find it, we need to first find the stationary vector  $\pi$ . By symmetry (or by noting that the chain is doubly stochastic),  $\pi_j = 1/n$ ,  $j = 1, \dots, n$ . Hence,

$$P_{ij}^* = \pi_j P_{ji} / \pi_i = P_{ji} = \begin{cases} p & \text{if } j = i - 1 \\ 1 - p & \text{if } j = i + 1 \end{cases}$$

(b) In general, the DTMC is not time reversible. It is in the special case  $p = 1/2$ . Otherwise, the probabilities of clockwise and counterclockwise motion are reversed.

**Problem 4.42** Imagine that there are edges between each of the pair of nodes  $i$  and  $i + 1$ ,  $i = 0, \dots, n - 1$ , and let the weight on edge  $(i, i + 1)$  be  $w_i$ , where

$$w_0 = 1$$

$$w_i = \prod_{j=1}^i \frac{p_j}{q_j}, \quad i \geq 1$$

where  $q_j = 1 - p_j$ . As a check, note that with these weights

$$P_{i,i+1} = \frac{w_i}{w_{i-1} + w_i} = \frac{p_i/q_i}{1 + p_i/q_i} = p_i, \quad 0 < i < n.$$

Since the sum of the weights on edges out of node  $i$  is  $w_{i-1} + w_i$ ,  $i = 1, \dots, n - 1$ , it follows that

$$\pi_0 = c$$

$$\pi_i = c \left[ \prod_{j=1}^{i-1} \frac{p_j}{q_j} + \prod_{j=1}^i \frac{p_j}{q_j} \right] = \frac{c}{q_i} \prod_{j=1}^{i-1} \frac{p_j}{q_j}, \quad 0 < i < n$$

$$\pi_n = c \prod_{j=1}^{n-1} \frac{p_j}{q_j}$$

where  $c$  is chosen to make  $\sum_{j=0}^n \pi_j = 1$ .

**Problem 4.46** (a) Yes, it is a Markov chain. It suffices to construct the transition matrix and verify that the process has the Markov property. Let  $P^*$  be the new transition matrix. Then we have, for  $0 \leq i \leq N$  and  $0 \leq j \leq N$ ,

$$P_{i,j}^* = P_{i,j} + \sum_{k=N+1}^{\infty} P_{i,k} B_{k,j}^{(N)},$$

where  $B_{k,j}^{(N)}$  is the probability of absorption into the absorbing state  $j$  in the absorbing Markov chain, where the states  $N + 1, N + 2, \dots$  are the transient states, while the state  $1, 2, \dots, N$  are the  $N$  absorbing states. In other words,  $B_{k,j}^{(N)}$  is the probability that the next state with index in the set  $\{1, 2, \dots, N\}$  visited by the Markov chain, starting with  $k > N$  is in fact  $j$ . It is easy to see that the markov property is still present.

(b) The proportion of time in  $j$  is  $\pi_j / \sum_{i=1}^N \pi_i$ .

(c) Let  $\pi_i(N)$  be the steady-state probabilities for the chain, only counting to visits among the states in the subset  $\{1, 2, \dots, N\}$ . (This chain is necessarily positive recurrent.) By renewal theory,

$$\pi_i(N) = (E[\text{Number of } Y - \text{transitions between } Y - \text{visits to } i])^{-1}$$

and

$$\begin{aligned}\pi_j(N) &= \frac{E[\text{No. } Y\text{-transitions to } j \text{ between } Y \text{ visits to } i]}{E[\text{No. } Y\text{-transitions to } i \text{ between } Y \text{ visits to } i]} \\ &= \frac{E[\text{No. } X\text{-transitions to } j \text{ between } X \text{ visits to } i]}{1/\pi_i(N)}\end{aligned}$$

(d) For the symmetric random walk, the new MC is doubly stochastic, so  $\pi_i(N) = 1/(N+1)$  for all  $i$ . By part (c), we have the conclusion.

(e) It suffices to show that

$$\pi_i(N)P_{i,j}^* = \pi_j(N)P_{j,i}^*$$

for all  $i$  and  $j$  with  $i \leq N$  and  $j \leq N$ . However, by above,

$$\pi_i(N)P_{i,j}^* = \pi_i(N)P_{i,j} + \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k}B_{k,j}^{(N)},$$

and

$$\pi_j(N)P_{j,i}^* = \pi_j(N)P_{j,i} + \pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k}B_{k,i}^{(N)},$$

The two terms on the right are equal in these two displays. First, by the original reversibility, we have

$$\pi_i(N)P_{i,j} = \pi_j(N)P_{j,i}.$$

Second, by Theorem 4.7.2, we have

$$\pi_j(N) \sum_{k=N+1}^{\infty} P_{j,k}B_{k,i}^{(N)} = \pi_i(N) \sum_{k=N+1}^{\infty} P_{i,k}B_{k,j}^{(N)}.$$

We see that by expanding into the individual paths, and seeing that there is a reverse path.

**Problem 4.47** Intuitively, in steady state each ball is equally likely to be in any of the urns and the positions of the balls are independent. Hence it seems intuitive that

$$\pi(\underline{n}) = \frac{M!}{n_1! \cdots n_m!} \left(\frac{1}{m}\right)^M.$$

To check the above and simultaneously establish time reversibility let

$$\underline{n}' = (n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_{j-1}, n_j + 1, n_{j+1}, \dots, n_m)$$

and note that

$$\begin{aligned}
\pi(\underline{n})P(\underline{n}, \underline{n}') &= \frac{M!}{n_1! \cdots n_m!} \left(\frac{1}{m}\right)^M \frac{n_i}{M} \frac{1}{m-1} \\
&= \frac{M!}{n_1! \cdots (n_i - 1)! \cdots (n_j + 1)! \cdots n_m!} \left(\frac{1}{m}\right)^M \frac{n_j + 1}{M} \frac{1}{m-1} \\
&= \pi(\underline{n}')P(\underline{n}', \underline{n}).
\end{aligned}$$

**Problem 4.48 (a)** Each transition into  $i$  begins a new cycle. A reward of 1 is earned if state visited from  $i$  is  $j$ . Hence average reward per unit time is  $P_{ij}/\mu_{ii}$ .

(b) Follows from (a) since  $1/\mu_{jj}$  is the rate at which transitions into  $j$  occur.

(c) Suppose a reward rate of 1 per unit time when in  $i$  and heading for  $j$ . New cycle whenever enter  $i$ . Hence, average reward per unit time is  $P_{ij}\eta_{ij}/\mu_{ii}$ .

(d) Consider (c) but now only give a reward at rate 1 per unit time when the transition time from  $i$  to  $j$  is within  $x$  time units. Average reward is

$$\begin{aligned}
\frac{\mathbf{E}[\text{Reward per cycle}]}{\mathbf{E}[\text{Time of cycle}]} &= \frac{P_{ij}\mathbf{E}[\min(X_{ij}, x)]}{\mu_{ii}} \\
&= \frac{P_{ij} \int_0^x \bar{F}_{ij}(y) dy}{\mu_{ii}} \\
&= \frac{P_{ij}\eta_{ij}F_{ij}^c(x)}{\mu_{ii}}
\end{aligned}$$

where  $X_{ij} \sim F_{ij}$ .

**Problem 4.49**

$$\begin{aligned}
\lim_{t \rightarrow \infty} \mathbf{P}(S(t) = j | X(t) = i) &= \frac{\lim_{t \rightarrow \infty} \mathbf{P}(S(t) = j, X(t) = i)}{\mathbf{P}(X(t) = i)} \\
&= \frac{P_{ij} \int_0^\infty \bar{F}_{ij}(y) dy / \mu_{ii}}{P_i} \quad \text{by Theorem 4.8.4} \\
&= \frac{P_{ij}\eta_{ij}}{\mu_i}
\end{aligned}$$

**Problem 4.50**  $\pi = (6, 3, 5)/14$ ,  $\mu_1 = 25$ ,  $\mu_2 = 80/3$ , and  $\mu_3 = 30$ .

(a)

$$\begin{aligned}
P_1 &= \frac{6 \times 25}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{15}{38} \\
P_2 &= \frac{3 \times \frac{80}{3}}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{8}{38} \\
P_3 &= \frac{5 \times 30}{6 \times 25 + 3 \times \frac{80}{3} + 5 \times 30} = \frac{15}{38}
\end{aligned}$$

(b)

$$\text{P(heading for 2)} = P_1 \frac{P_{12}t_{12}}{\mu_1} = \frac{15}{38} \times \frac{10}{25} = \frac{3}{19}$$

(c)

$$\text{fraction of time from 2 to 3} = P_2 \frac{P_{23}t_{23}}{\mu_2} = \frac{8}{38} \times \frac{60}{80} = \frac{3}{19}$$