

# IEOR 6711: Stochastic Models I

Fall 2012, Professor Whitt

## Solutions to Homework Assignment 12 due on Thursday, November 29

**Problem 5.3 (a)** Let  $N(t)$  denote the number of transitions by  $t$ . It is easy to show in this case that

$$\mathbb{P}(N(t) \geq n) \leq \sum_{j=n}^{\infty} e^{-Mt} \frac{(Mt)^j}{j!}$$

and thus  $\mathbb{P}(N(t) < \infty) = 1$ .

**(b)** Let  $X_{n+1}$  denote the time between the  $n$ -th and  $(n+1)$ -st transition and let  $J_n$  denote the  $n$ -th state visited. Also if we let

$$N(t) \triangleq \sup\{n : X_1 + \cdots + X_n \leq t\}$$

then  $N(t)$  denotes the number of transitions by  $t$ . Now let  $j$  be the first recurrent state that is reached and suppose it was reached at the  $n_0$ -th transition ( $n_0$  must be finite by assumption). Let  $n_1, n_2, \dots$  be the successive integers  $n$  at which  $J_n = j$ . (Such integers exist since  $j$  is recurrent.) Set  $T_0 = X_1 + \cdots + X_{n_0}$ , and

$$T_k \triangleq X_{n_{k-1}+1} + \cdots + X_{n_k} .$$

In other words,  $T_k$  denote the amount of time between the  $k$ -th and  $(k+1)$ -th visit to  $j$ . Therefore it follows that  $\{T_k, k \geq 1\}$  forms a renewal process, and so  $\sum_{k=1}^{\infty} T_k = \infty$  with probability 1. Since

$$\sum_{n=1}^{\infty} X_n = \sum_{k=0}^{\infty} T_k$$

it follows that  $\sum_{n=1}^{\infty} X_n = \infty$ .

**Problem 5.4** Let  $T_i$  denote the time to go from  $i$  to  $i+1$ ,  $i \geq 0$ . Then  $\sum_{i=0}^{N-1} T_i$  is the time to go from 0 to  $N$ . Now  $T_i$  is exponential with rate  $\lambda_i$  and the  $T_i$  are independent. Hence

$$\mathbb{E} \left[ e^{s \sum_{i=0}^{N-1} T_i} \right] = \prod_{i=0}^{N-1} \frac{\lambda_i}{\lambda_i - s} .$$

We may use it to compute the mean and variance or we can do directly and mean =  $1/\lambda_0 + \cdots + 1/\lambda_{N-1}$ , variance =  $1/\lambda_0^2 + \cdots + 1/\lambda_{N-1}^2$ .

**Problem 5.9**

$$\begin{aligned}
P_{ij}(t+s) &= \sum_k P(X(t+s) = j \mid X_0 = i, X(t) = k)P(X(t) = k \mid X_0 = i) \\
&= \sum_k P_{kj}(s)P_{ik}(t) .
\end{aligned}$$

**Problem 5.10 (a)**

$$\lim_{t \rightarrow 0} \frac{1 - P(t)}{t} = v_0 .$$

(b) The first inequality follows from exercise 5.9 and

$$\begin{aligned}
P(t+s) &= P(X(t+s) = 0 \mid X(0) = 0, X(s) = 0)P(s) \\
&\quad + P(X(t+s) = 0 \mid X(0) = 0, X(s) \neq 0)(1 - P(s)) \\
&\leq P(t)P(s) + 1 - P(s) .
\end{aligned}$$

(c) From (b)

$$P(s)P(t-s) \leq P(t) \leq P(s)P(t-s) + 1 - P(t-s)$$

or

$$P(s) + P(t-s) - 1 \leq P(t) \leq P(s) + 1 - P(t-s)$$

where the left hand inequality follows from

$$P(s)(1 - P(t-s)) \leq 1 - P(t-s) .$$

$\lim_{s \rightarrow t} P(s-t) = 1$  implies the continuity of  $P$ .

**Problem 5.13**

$$\prod_{j=i}^{i+k-1} \frac{\lambda_j}{\lambda_j + \mu_j} .$$

**Problem 5.15 (a)** Birth and death process.

(b)  $\lambda_n = n\lambda + \theta$ ,  $\mu_n = n\mu$ .

(c) Set  $M(t) = E[X(t) \mid X(0) = i]$ . Then

$$E[X(t+h) \mid X(t)] = X(t) + (\lambda X(t) + \theta)h - \mu X(t)h + o(h)$$

and so  $M(t+h) = M(t) + (\lambda - \mu)M(t)h + \theta h + o(h)$ . Therefore,  $M'(t) = (\lambda - \mu)M(t) + \theta$  or  $e^{-(\lambda - \mu)t}[M'(t) - (\lambda - \mu)M(t)] = \theta e^{-(\lambda - \mu)t}$ . Integrating both sides gives

$$e^{-(\lambda - \mu)t}M(t) = -\frac{\theta}{\lambda - \mu}e^{-(\lambda - \mu)t} + C$$

or

$$M(t) = Ce^{-(\lambda-\mu)t} - \frac{\theta}{\lambda - \mu} .$$

As  $M(0) = i$  we obtain

$$M(t) = \frac{\theta}{\lambda - \mu} \left( e^{-(\lambda-\mu)t} - 1 \right) + ie^{-(\lambda-\mu)t} .$$

**Problem 5.21** With the number of customers in the shop as the state, we get a birth and death process with  $\lambda_0 = \lambda_1 = 3$ ,  $\mu_1 = \mu_2 = 4$ . Therefore  $P_1 = \frac{3}{4}P_0$ ,  $P_2 = \frac{3}{4}P_1 = \left(\frac{3}{4}\right)^2 P_0$ . And since  $P_0 + P_1 + P_2 = 1$ , we get  $P_0 = 16/37$ .

(a)

$$P_1 + 2P_2 = \left[ \frac{3}{4} + 2 \left( \frac{3}{4} \right)^2 \right] P_0 = \frac{30}{37}$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{37} = \frac{28}{37} .$$

(c)  $\mu = 8$ , and so  $P_0 = \frac{64}{97}$ . So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left( \frac{3}{8} \right)^2 \frac{64}{97} = \frac{88}{97} .$$

The rate of added customers is therefore

$$\lambda \frac{88}{97} - \lambda \frac{28}{37} \simeq 0.45 .$$

The business he does would improve by 0.45 customers per hour.