

# IEOR 6711: Stochastic Models I

Fall 2012, Professor Whitt

## Solutions to Homework Assignment 9 due on Thursday, November 8

**Problem 4.1** Let  $D_n$  be the random demand of time period  $n$ . Clearly  $D_n$  is i.i.d. and independent of all  $X_k$  for  $k < n$ . Then we can represent  $X_{n+1}$  by

$$X_{n+1} = \max\{0, X_n \cdot \mathbf{1}_{[s, \infty)}(X_n) + S \cdot \mathbf{1}_{[0, s)}(X_n) - D_{n+1}\}$$

which depends only on  $X_n$  since  $D_{n+1}$  is independent of all history. Hence  $\{X_n, n \geq 1\}$  is a Markov chain. It is easy to see assuming  $\alpha_k = 0$  for  $k < 0$ ,

$$P_{ij} = \begin{cases} \alpha_{S-j} & \text{if } i < s, j > 0 \\ \sum_{k=S}^{\infty} \alpha_k & \text{if } i < s, j = 0 \\ \alpha_{i-j} & \text{if } i \geq s, j > 0 \\ \sum_{k=i}^{\infty} \alpha_k & \text{if } i \geq s, j = 0 \end{cases}$$

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The following three problems (4.2, 4.4, 4.5) needs a fact:

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)$$

which requires a proof to use. Try to prove it by yourself.

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**Problem 4.2** Let  $\mathcal{S}$  be the state space. First we show that

$$\mathbb{P}(X_{n_k+1} = j | X_{n_1} = i_1, \dots, X_{n_k} = i_k) = \mathbb{P}(X_{n_k+1} = j | X_{n_k} = i_k)$$

by the following : Let  $A = \{X_{n_k+1} = j\}$ ,  $B = \{X_{n_1} = i_1, \dots, X_{n_k} = i_k\}$  and  $B_b, b \in \mathcal{I}$  are elements of  $\{(X_l, l \leq n_k, l \neq n_1, \dots, l \neq n_k) : X_l \in \mathcal{S}\}$ .

$$\begin{aligned} \mathbb{P}(A|B) &= \sum_{b \in \mathcal{I}} \mathbb{P}(A \cap B_b|B) \\ &= \sum_{b \in \mathcal{I}} \mathbb{P}(A|B_b \cap B)\mathbb{P}(B_b|B) \\ &= \sum_{b \in \mathcal{I}} \mathbb{P}(A|X_{n_k} = i_k)\mathbb{P}(B_b|B) \\ &= \mathbb{P}(A|X_{n_k} = i_k) \sum_{b \in \mathcal{I}} \mathbb{P}(B_b|B) \\ &= \mathbb{P}(A|X_{n_k} = i_k)\mathbb{P}(\Omega|B) \\ &= \mathbb{P}(X_{n_k+1} = j | X_{n_k} = i_k) . \end{aligned}$$

We consider the mathematical induction on  $l \equiv n - m$ . For  $l = 1$ , we just showed. Now assume that the statement is true for all  $l \leq l^*$  and consider  $l = l^* + 1$ :

$$\begin{aligned}
& \mathbb{P}(X_n = j | X_{n_1} = i_1, \dots, X_{n_k} = i_k) \\
&= \sum_{i \in \mathcal{S}} \mathbb{P}(X_n = j, X_{n-1} = i | X_{n_1} = i_1, \dots, X_{n_k} = i_k) \\
&= \sum_{i \in \mathcal{S}} \mathbb{P}(X_n = j | X_{n-1} = i, X_{n_1} = i_1, \dots, X_{n_k} = i_k) \mathbb{P}(X_{n-1} = i | X_{n_1} = i_1, \dots, X_{n_k} = i_k) \\
&= \sum_{i \in \mathcal{S}} \mathbb{P}(X_n = j | X_{n-1} = i) \mathbb{P}(X_{n-1} = i | X_{n_k} = i_k) \quad \text{By } l \leq l^* \text{ cases} \\
&= \sum_{i \in \mathcal{S}} \mathbb{P}(X_n = j | X_{n-1} = i, X_{n_k} = i_k) \mathbb{P}(X_{n-1} = i | X_{n_k} = i_k) \\
&= \sum_{i \in \mathcal{S}} \mathbb{P}(X_n = j, X_{n-1} = i | X_{n_k} = i_k) \\
&= \mathbb{P}(X_n = j | X_{n_k} = i_k)
\end{aligned}$$

which completes the proof for  $l = l^* + 1$  case.

**Problem 4.3** Simply by *Pigeon hole principle* which saying that if  $n$  pigeons return to their  $m (< n)$  home (through hole), then at least one home contains more than one pigeon.

Consider any path of states  $i_0 = i, i_1, \dots, i_n = j$  such that  $P_{i_k, i_{k+1}} > 0$ . Call this a path from  $i$  to  $j$ . If  $j$  can be reached from  $i$ , then there must be a path from  $i$  to  $j$ . Let  $i_0, \dots, i_n$  be such a path. If all of values  $i_0, \dots, i_n$  are not distinct, then there must be a subpath from  $i$  to  $j$  having fewer elements (for instance, if  $i, 1, 2, 4, 1, 3, j$  is a path, then so is  $i, 1, 3, j$ ). Hence, if a path exists, there must be one with all distinct states.

**Problem 4.4** Let  $Y$  be the first passage time to the state  $j$  starting the state  $i$  at time 0.

$$\begin{aligned}
P_{ij}^n &= \mathbb{P}(X_n = j | X_0 = i) \\
&= \sum_{k=0}^n \mathbb{P}(X_n = j, Y = k | X_0 = i) \\
&= \sum_{k=0}^n \mathbb{P}(X_n = j | Y = k, X_0 = i) \mathbb{P}(Y = k | X_0 = i) \\
&= \sum_{k=0}^n \mathbb{P}(X_n = j | X_k = j) \mathbb{P}(Y = k | X_0 = i) \\
&= \sum_{k=0}^n P_{jj}^{n-k} f_{ij}^k
\end{aligned}$$

**Problem 4.5 (a)** The probability that the chain, starting in state  $i$ , will be in state  $j$  at time  $n$  without ever having made a transition into state  $k$ .

- (b) Let  $Y$  be the last time leaving the state  $i$  before first reaching to the state  $j$  starting the state  $i$  at time 0.

$$\begin{aligned}
P_{ij}^n &= P(X_n = j | X_0 = i) \\
&= \sum_{k=0}^n P(X_n = j, Y = k | X_0 = i) \\
&= \sum_{k=0}^n P(X_n = j, Y = k, X_k = i | X_0 = i) \\
&= \sum_{k=0}^n P(X_n = j, Y = k | X_k = i, X_0 = i) P(X_k = i | X_0 = i) \\
&= \sum_{k=0}^n P(X_n = j, Y = k | X_k = i) P_{ii}^k \\
&= \sum_{k=0}^n P(X_n = j, X_l \neq i, l = k + 1, \dots, n - 1 | X_k = i) P_{ii}^k \\
&= \sum_{k=0}^n P_{ij/i}^{n-k} P_{ii}^k
\end{aligned}$$

**Problem 4.7**

- (a)  $\infty$

Here is an argument: Let  $x$  be the expected number of steps required to return to the initial state (the origin). Let  $y$  be the expected number of steps to move to the left 2 steps, which is the same as the expected number of steps required to move to the right 2 steps. Note that the expected number of steps required to go to the left 4 steps is clearly  $2y$ , because you first need to go to the left 2 steps, and from there you need to go to the left 2 steps again. Then, consider what happens in successive pairs of steps: Using symmetry, we get

$$x = 2 + (0 \times (1/2) + y \times (1/2)) = 2 + y/2$$

and

$$y = 2 + (0 \times (1/4) + y \times (1/2) + (2 * y) \times (1/4))$$

If we subtract  $y$  from both sides, this last equation yields

$$2 = 0 .$$

Hence there is no finite solution. The quantity  $y$  must be infinite; a finite value cannot solve the equation.

- (b) Note that the expected number of returns in  $2n$  steps is the sum of the probabilities of returning in  $2k$  steps for  $k$  from 1 to  $n$ , each term of which is binomial. Thus, we have

$$E[N_{2n}] = \sum_{k=1}^n \frac{(2k)!}{k!k!} (1/2)^{2k} ,$$

which can be shown to be equal to the given expression by mathematical induction.

(c) We say that  $f(n) \sim g(n)$  as  $n \rightarrow \infty$  if

$$f(n)/g(n) \rightarrow 1 \quad \text{as } n \rightarrow \infty .$$

By Stirling's approximation,

$$(2n+1) \frac{(2n)!}{n!n!} (1/2)^{2n} \sim 2\sqrt{n/\pi} ,$$

so that

$$E[N_n] \sim \sqrt{2n/\pi} \quad \text{as } n \rightarrow \infty .$$

**Problem 4.8 (a)**

$$P_{ij} = \frac{\alpha_j}{\sum_{k=i+1}^{\infty} \alpha_k} , \quad j > i$$

(b)  $\{T_i, i \geq 1\}$  is not a Markov chain - the distribution of  $T_i$  does depend on  $R_i$ .  $\{(R_{i+1}, T_i), i \geq 1\}$  is a Markov chain.

$$\begin{aligned} P(R_{i+1} = j, T_i = n | R_i = l, T_{i-1} = m) &= \frac{\alpha_j}{\sum_{k=l+1}^{\infty} \alpha_k} \left( \sum_{k=0}^l \alpha_k \right)^{n-1} \sum_{k=l+1}^{\infty} \alpha_k \\ &= \alpha_j \left( \sum_{k=0}^l \alpha_k \right)^{n-1} , \quad j > l \end{aligned}$$

(c) If  $S_n = j$  then the  $(n+1)^{st}$  record occurred at time  $j$ . However, knowledge of when these  $n+1$  records occurred does not yield any information about the set of values  $\{X_1, \dots, X_j\}$ . Hence, the probability that the next record occurs at time  $k$ ,  $k > j$ , is the probability that both  $\max\{X_1, \dots, X_j\} = \max\{X_1, \dots, X_{k-1}\}$  and that  $X_k = \max\{X_1, \dots, X_k\}$ . Therefore, we see that  $\{S_n\}$  is a Markov chain with

$$P_{jk} = \frac{j}{k-1} \frac{1}{k} , \quad k > j .$$

**Problem 4.11 (a)**

$$\begin{aligned} \sum_{n=1}^{\infty} P_{ij}^n &= E[\text{number of visits to } j | X_0 = i] \\ &= E[\text{number of visits to } j | \text{ever visit } j, X_0 = i] f_{ij} \\ &= (1 + E[\text{number of visits to } j | X_0 = j]) f_{ij} \\ &= \frac{f_{ij}}{1 - f_{jj}} < \infty . \end{aligned}$$

since  $1 + \text{number of visits to } j | X_0 = j$  is geometric with mean  $\frac{1}{1-f_{jj}}$ .

(b) Follows from above since

$$\begin{aligned}\frac{1}{1 - f_{jj}} &= 1 + \mathbf{E}[\text{number of visits to } j | X_0 = j] \\ &= 1 + \sum_{n=1}^{\infty} P_{jj}^n .\end{aligned}$$

**Problem 4.12** If we add the irreducibility of  $\mathbf{P}$ , it is easy to see that  $\boldsymbol{\pi} = \frac{1}{n}\mathbf{1}$  is a (and the unique) limiting probability.