IEOR 6711: Stochastic Models I

Fall 2012, Professor Whitt

Topic for Discussion, Tuesday, October 2

Limits for Counting and Compound Processes

1. Counting Processes and the Inverse Relation

A counting process is defined in §2.1 of Ross, right at the beginning. It is a generalization of a renewal (counting) process, which in turn is a generalization of a Poisson (counting) process, typically denoted by $\{N(t) : t \geq 0\}$. For a counting process, there are no specific stochastic (distributional) assumptions. The random variable $N(t)$ counts the number of events or points appearing in the time interval $[0, t]$.

A counting process has an alternative representation through the sequence of partial sums of the intervals between successive points. This structure is used to discuss a renewal process in §3.1 of Ross, right at the beginning. There is an equivalence (one-to-one representation) between the stochastic processes $\{S_n : n \geq 0\}$ and $\{N(t) : t \geq 0\}$, characterized by the **inverse relation**

$$S_n \leq t \quad \text{if and only if} \quad N(t) \geq n, \tag{1}$$

which is given in (3.2.1) on page 99 of Ross.

This inverse relation is easy to view from a plot of typical sample paths of the stochastic processes $\{S_n : n \geq 0\}$ and $\{N(t) : t \geq 0\}$. If we plot a sample path of $\{N(t) : t \geq 0\}$, we have t on the x axis and $N(t)$ on the y axis. At the same time, i.e., in the same plot, we have n on the y axis and S_n on the x axis. Thus this inverse relation is naturally expressed as a mapping of one sample path (a function) into the other sample path (another function). It thus natural to view the relation as a map from a space of functions (sample paths) to itself.

2. Application of the Inverse Relation in Renewal Theory

That equivalence relation (1) (inverse relation) is used to find an explicit expression for the probability mass function $P(N(t) = n)$ for a renewal process on p. 99 of Ross:

$$P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n + 1) = P(S_n \leq t) - P(S_{n+1} \leq t).$$

It is used to find an explicit representation for the **renewal function**

$$m(t) \equiv E[N(t)] = \sum_{n=1} P(N(t) \geq n) = \sum_{n=1} P(S_n \leq t);$$

see Exercise 1.1 and Proposition 3.2.1. Thus, if the time between renewals is distributed as a random variable X with pdf $f(x)$ and Laplace transform

$$\hat{f}(s) \equiv E[e^{-sX}] = \int_0^\infty e^{-sx} f(x)\, dx,$$

then the Laplace transform of $m(t)$ is

$$\hat{m}(s) \equiv \int_0^\infty e^{-st} m(t)\, dt = \sum_{n=1} \frac{\hat{f}(s)^n}{s} = \frac{\hat{f}(s)}{s(1 - \hat{f}(s))}.$$

(We use the fact that if F is the cdf of the pdf f, then $\hat{F}(s) = \hat{f}(s)/s$.)

3. Application of the Inverse Relation to Prove Limit Theorems

The natural function space is the so-called function space $D \equiv D([0,\infty),\mathbb{R})$ of right-continuous real-valued functions on the interval $[0,\infty)$ with left limits everywhere, as discussed in my book; available on line:

partial book: http://www.columbia.edu/ ww2040/book.html.

full book: http://www.columbia.edu/ ww2040/jumps.html.

In particular, for more on the space D, see §3.3 of my book, but also §1.2. See the book *Convergence of Probability Measures* by P. Billingsley (1968, 1999) for more. (The following material mostly concerns §§7.3 and 7.4 of my book.)

Things are a little complicated in the setting of D. The sample path of $\{N(t) : t \geq 0\}$ is directly in the space D, or even the subset of nondecreasing piecewise-constant functions in D, but the inverse process is not so clear. However, we can associate with the sequence $\{S_n : n \geq 0\}$ a continuous time process, denoted by $\{S_{\lfloor t \rfloor} : t \geq 0\}$, where $\lfloor t \rfloor$ is the "floor function," i.e., the greatest integer less than or equal to t. That directly gives us a right-continuous stochastic process, but it does not coincide with what we see when we look at the plot in reverse, as a map from the y axis to the x axis. But the essential connection is there. See §§13.6-13.8 of my book for more discussion. To understand, it is good to contrast the CLT with the associated functional CLT (FCLT): A CLT might state

$$\frac{S_n - mn}{\sqrt{n\sigma^2}} \Rightarrow N(0,1) \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad n \to \infty,$$

whereas the associated FCLT states

$$\{\frac{S_{\lfloor nt \rfloor} - mnt}{\sqrt{n\sigma^2}} : t \geq 0\} \Rightarrow \{B(t) : t \geq 0\} \quad \text{in} \quad D \quad \text{as} \quad n \to \infty,$$

where $\Rightarrow$ is convergence in distribution and $\{B(t) : t \geq 0\}$ is standard Brownian motion.

4. SLLN's for Counting Processes

The SLLN for a renewal process is given in Proposition 3.3.1 on p. 102 of Ross. It follows from the previous SLLN for the associated partial sums. The IID conditions in the renewal process are only used to establish the SLLN of the partial sums. There is actually an equivalence between the limits, without imposing the IID conditions.

Theorem 0.1 (*equivalence of SLLN's*) *The limits $S_n/n \to a > 0$ as $n \to \infty$ w.p.1. and $N(t)/t \to 1/a > 0$ as $t \to \infty$ are equivalent.*

This result is a special case of Corollary 3.4.1 in the Internet Supplement of my book:

internet supplement to book: http://www.columbia.edu/ ww2040/supplement.html

4. CLT's for Counting Processes

There is a corresponding equivalence between CLT's for the two stochastic processes $\{S_n : n \geq 0\}$ and $\{N(t) : t \geq 0\}$, but the proof is actually somewhat complicated. First, we note that the CLT for a renewal counting process is given in Theorem 3.3.5 of Ross. The equivalence is discussed in Section 7.3 of my book. In particular, Theorem 7.3.1 on page 234 is a precise statement. This theorem also appears as Theorem 3.5.1 of the Internet Supplement. A special case of the regular varying function assumed there is $\psi(t) \equiv t^p$ for $0 < p < 1$. With the usual IID and finite-second-moment conditions, we have $p = 1/2$. This stated CLT equivalence extends a result in Glynn and Whitt (1988); see the book. To repeat, the proof is somewhat complicated. But something like that argument is used to prove Theorem 3.3.5 of Ross. The idea is that we use the CLT for S_n rather than the assumptions required to get that CLT.

A good application of the functions space framework is the CLT for the counting process corresponding to a superposition of independent renewal processes. This could be the total arrival process to a queue that comes from several independent sources. See §9.4 of the book.

5. Random Sums: Compound-Poisson Processes and Renewal-Reward Processes

A compound Poisson process is a stochastic process $\{R(t) : t \geq 0\}$, defined by

$$R(t) \equiv \sum_{k=1}^{N(t)} Y_k, \quad t \geq 0,$$

where $\{Y_k : k \geq 1\}$ is a sequence of IID random variables, which is independent of a Poisson process $\{N(t) : t \geq 0\}$; see §2.5 of Ross. We proved a CLT for a compound Poisson process in the last class.

If, instead, the process $\{N(t) : t \geq 0\}$ is a renewal process, then we obtain a renewal-reward process; then we interpret the random variables Y_k as rewards; see §3.6 of Ross. We can establish LLN's and CLT's for such compound processes. These too do not depend critically on the stochastic assumptions. Instead, we can start from established limits, however obtained, and then obtain the desired limits.

The SLLN for the renewal-reward process is given in Theorem 3.6.1 of Ross, drawing on Theorem 3.3.1. The proofs are very simple. Note that the independence conditions are only used to establish the first limit, which in turn implies the other limit.

As mentioned above, a CLT for the special case of a compound Poisson process was established last time in class. There are corresponding CLT's for renewal-reward stochastic processes and more general compound processes, but these are much more complicated.

These CLT's can also be established starting from the existence of limits, rather than from specific stochastic assumptions. e.g., see Section 7.4 of my book. The results there are stated with stochastic assumptions. Corollary 13.3.2 is a statement without specific stochastic assumptions. The application to random sums is discussed at the bottom of page 433 there.

These limits can advantageously be viewed as consequences of the continuous mapping theorem in the context of the function space D; see §3.4 of my book. The relevant continuous map is composition; see §§13.2-13.3 of my book. For a very nice account, see pages 151-156 of Billingsley, *Convergence of Probability Measures*, second edition, 1999. The role of uniform convergence is shown by the argument on lines 3-4 of p. 152.

A whole book devoted to this random sum issue is Gut, *Stopped Random Walks, Limit Theorems and Applications*, 1988.