

## IEOR 6711: Stochastic Models I

Fall 2012, Professor Whitt, Tuesday, September 25

### Poisson Process: A Special Case of Several Processes

It is useful to be aware that a Poisson process is a special case of several important stochastic processes. That leads to different equivalent definitions of a Poisson process, as in Definitions 2.1.1 and 2.1.2 of the Ross text. It also leads to different ways to analyze a Poisson process.

#### (a) Poisson random measure

**Definition 2.1.1** can be viewed as a special case of a Poisson process defined as a Poisson random measure (on a subset of a Euclidean space  $\mathbb{R}^k$  with an intensity function  $\lambda(x)$ , where  $k = 1$  and the subset in  $[0, \infty)$  and  $\lambda(x) \equiv \lambda$  for some positive constant  $\lambda$ ). A Poisson process (as well as a nonhomogeneous Poisson process - Section 2.4 - can be viewed as a special case of a Poisson random measure. In the standard case, the underlying space is the positive real line  $[0, \infty)$ . But Poisson random measures can be defined on more general spaces, such as  $\mathbb{R}^2$ , corresponding to random points on the blackboard. Exercise 2.33 discusses a special case of a Poisson random measure in which the space is  $\mathbb{R}^2$ .

#### (b) CTMC

**Definition 2.1.2** can be viewed as a special case of a Poisson process defined as a special case of a continuous-time Markov chain (CTMC). One way to characterize a CTMC is via its infinitesimal rate matrix, usually denoted by  $Q$ . For a Poisson process, we have  $Q_{i,i+1} = \lambda$ ,  $Q_{i,i} = -\lambda$  and  $Q_{i,j} = 0$  for all other  $j$ . The rate matrix  $Q$  determines the probability transition matrix  $P(t) \equiv (P_{i,j}(t))$  via a matrix ordinary differential equation (ODE)

$$\dot{P}(t) = P(t)Q = QP(t) ;$$

see the CTMC lecture notes posted on line for the end of the course in November. That corresponds to Definition 2.1.2. Notice that the "little oh" notation in (iii) of definition 2.1.2 just means that the function has a derivative (from the right at 0).

#### (c) Lévy Process

A Poisson Process can also be viewed a special case of a Lévy process. A Lévy process is a stochastic process with stationary and independent increments. Assume that it takes the value 0 at time 0. (This is another way to look at Definition 2.1.2.) The unique Lévy process with continuous sample paths is Brownian motion. (You do not need to directly assume that the increments have a normal distribution.) The unique Lévy process with sample paths having only unit jumps is the Poisson process. (You do not need to directly assume that the increments have a Poisson distribution.) The general Lévy process can be constructed from an independent Brownian motion and Poisson processes. This corresponds to Definition 2.1.1 (which is not stated in such a minimal elegant way).

#### (d) renewal process

A Poisson process is a special case of a renewal process (Chapter 3) in which the times between renewals have an exponential distribution. This corresponds to Proposition 2.2.1.

### 2. Basic Properties

Please pay attention to the basic properties on the Concise Summary page, such as Theorem 2.3.1 and Proposition 2.3.2.