There are four questions, each with several parts. Questions 2 and 4 are longer than the others and thus count more.

1. The Eight (8) Subway Line. (20 points)

A new subway line has been added to the West Side for the convenience of Columbia students. It has six stations. Going north, it starts at 88th street (station 1) and has stops at 98th street (station 2), 108th street (station 3), 118th street (station 4), 128th street (station 5) and 138th street (station 6). It can change tracks and directions at the two end points, so that the trains travel in a loop, going north from station 1 to station 6 and then back south from station 6 to station 1, where it then goes north again. Subway trains follow a strict schedule: The travel time between successive stations is constant, equal to 2 minutes. There are two subway trains, one starting north from station 1 and the other starting south from station 6. Thus, at station 2, the intervals between successive trains in a specified direction are exactly 10 minutes. That is, a southbound train comes to station 2 every 10 minutes and also a northbound train comes to station 2 every 10 minutes. At the end stations, the story is different; e.g., at station 1 all arriving subway trains are southbound from 2, but one arrives every 10 minutes.

Customers arrive at station i to use the subway according to a Poisson process with rate λ_i per minute. Suppose that the subway has unlimited capacity and that the time to load and unload passengers can be ignored. Suppose that each customer entering station i gets off at station j with probability $P_{i,j}$, independently of all other customers (where $P_{i,i} = 0$). Suppose that people get on subways only in the direction they want to go.

(a) Give an expression for the expected number of customers to get on the subway (necessarily going north) at each visit to station 1.

(b) Suppose that 8 customers get on the subway at station 1 (necessarily going north) at time t. What is the probability that exactly 3 of these customers had to wait more than 4 minutes before getting on the subway?

(c) Give an expression for the probability that the number of customers getting off the northbound subway at a visit to station 4 is exactly j.

(d) Give an expression for the probability that, simultaneously, the number of customers getting off the northbound subway at a visit to station 4 is j and the number getting off at the next stop, at station 5, is k.

(e) Suppose that $\lambda_i = 2$ for all i and $P_{i,j} = 1/5$ for all j with $j \neq i$. How can you determine a convenient accurate approximation for the probability that the number of customers getting off the northbound subway at one specified visit to station 5 is greater than 20? Is that probability more than $1/20$?

2. The Movement of a Taxi (30 points)

A continuously operating taxi serves three locations: A, B and C.

idle times:
The taxi sits idle at each location an exponential length of time before departing to make a trip to one of the other two locations. The mean idle times are 2 minutes at A, 1 minute at B and 2 minutes at C. The idle times and travel times are mutually independent.

transition probabilities:
From A, the taxi next goes to B with probability $1/3$ and to C with probability $2/3$.
From B, the taxi next goes to A with probability $1/2$ and to C with probability $1/2$.
From C, the taxi next goes to B with probability $1/3$ and to A with probability $2/3$.

travel times:
The travel times between A and B in either direction are uniformly distributed in the interval $[5, 15]$ minutes.
The travel times between A and C in either direction are uniformly distributed in the interval $[20, 60]$ minutes.
The travel times between B and C in either direction are uniformly distributed in the interval $[20, 40]$ minutes.

(a) What is the long-run proportion of all taxi trips starting from location A?

(b) What is the long-run proportion of time that the taxi's most recent stop was at location A?

(c) What is the long-run proportion of time that the taxi is idle at location A?

(d) Let $P_t(A)$ be the probability that the taxi is idle at location A at time t. Does $P_t(A)$ converge to a proper limit as $t \to \infty$? Why or why not? If so, what is that limit?

(e) What is the rate at which the taxi makes trips departing from location A heading toward location B?

(f) What is the long-run conditional probability that the taxi will come next to location B, given that the taxi is now traveling away from location A?

(g) What is the long-run proportion of time that the taxi is traveling from A to C and the remaining time before getting to C is at least 30 minutes?

(h) Which of the previous answers would change if the travel times were changed from uniform to exponential with the same mean? (You need not do any new computations?)

(i) Suppose that the travel times are indeed changed from uniform to exponential with the same mean. Let $X(t)$ be the state of the taxi at time t, e.g., idle at A or traveling from A to B. Give an explicit formula (not numerical value) for the conditional probability

$$P(X(2) = \text{idle at } B \quad \text{and} \quad X(7) = \text{idle at } C | X(0) = \text{idle at } A).$$

3. Uniform random numbers (20 points)

Consider a sequence of i.i.d. uniform random numbers $\{U_n : n \geq 1\}$, where U_n is uniformly distributed on the interval $[0, 1]$. Let S_n be the sum of the first n uniform numbers, i.e.,

$$S_n \equiv U_1 + U_2 + \cdots + U_n, \quad n \geq 1,$$

with $S_0 \equiv 0$. Let F_n be the *fractional part* of S_n, defined by

$$F_n \equiv S_n - \lfloor S_n \rfloor, \quad n \geq 1,$$

where $\lfloor x \rfloor$ is the floor function, yielding the greatest integer less than or equal to the real number x. Let R_n be the *remainder* beyond n of the first partial sum to exceed n. That is let Z_n be the least integer k such that $S_k > n$, and let

$$R_n \equiv S_{Z_n} - n, \quad n \geq 1.$$

Let $\Rightarrow$ denote convergence in distribution.

    (a) Is the stochastic process $\{F_n : n \geq 1\}$ a Markov process? Why or why not?

    (b) Is the stochastic process $\{R_n : n \geq 1\}$ a Markov process? Why or why not?

    (c) Prove that there exists a random variable F such that $F_n \Rightarrow F$ as $n \to \infty$ and determine the probability distribution of the random variable F.

    (d) Prove that there exists a random variable R such that $R_n \Rightarrow R$ as $n \to \infty$ and determine the probability distribution of the random variable R.

    (e) Compare the probability distributions of the random variables F and R. Are the distributions the same? Are the distributions stochastically ordered? Or do neither of these relations hold?

4. New Airport Security Check (30 points)

A new elaborate airport security check has been designed with three inspection stations. At each inspection station, passengers are processed one at a time in order of arrival at that station. There is ample waiting space at each station. Suppose that the processing times at the stations are exponentially distributed random variables. Let the mean processing times be 10 seconds at station 1, 20 second at station 2 and 10 minutes at station 3 (more serious inspection).

Suppose that passengers may enter the security check system at either station 1 or station 2. Suppose that passengers arrive at station 1 according to a Poisson process with rate 2 per minute; suppose that passengers arrive at station 2 according to a Poisson process with rate 1 per minute.

Suppose that 1/4 of all passengers undergoing inspection at station 1 must repeat inspection at station 1, where they are required to go to the end of the queue at station 1. Suppose that 1/2 of all passengers undergoing inspection at station 1 must go next to inspection at station 2, where they are required to go to the end of the queue at station 2. Suppose that 1/100 of all passengers completing inspection at station 1 must go next to inspection at station 3. Suppose that 1/2 of all passengers undergoing inspection at station 2 must go next to inspection at station 1, where they are required to go to the end of the queue at station 1. Suppose that no customers completing inspection at station 2 need to immediately repeat inspection at station 2. Suppose that 1/50 of all passengers completing inspection at station 2 must go next to

inspection at station 3. The remaining passengers completing inspection at stations 1 and 2 leave the system. All passengers completing inspection at station 3 leave the system after completing inspection at station 3. Suppose that 1/1000 passengers completing inspection at station 3 are classified as a serious security risk.

(a) Specify the customary (required) assumptions on the model elements that make the stochastic process recording the number of passengers at each of the three stations a continuous-time Markov chain (CTMC), and specify the model. Henceforth assume that these assumptions are in force.

(b) Given this model, what is the long-run proportion of time that station 2 is busy? What is the long-run proportion of time that station 1 is busy? What is the long-run average waiting time per passenger to complete the entire inspection process.

(c) How do long-run proportions of time that stations 1 and 2 are busy in part (b) change when the arrival rate of passengers at station 1 from outside the system is changed from 2 per minute to 1 per minute? Henceforth (for all the remaining questions), assume that the arrival rate at station 1 from outside the system is indeed 1 per minute.

(d) Given the adjusted model in part (c), what is the probability that there is simultaneously 1 passenger at station 1, 2 passengers at station 2 and 3 passengers at station 3 at some time t after the system has been operating for a long time?

(e) Given the model, what is the long-run proportion of all arriving passengers that will be classified as a serious security risk?

(f) If possible, construct the reverse-time Markov chain associated with the CTMC specified in part (a), as adjusted in part (c).

(g) Identify conditions, if possible, under which the stochastic process recording the number of passengers at each of the three stations is a time-reversible continuous-time Markov chain.

(h) Indicate how to efficiently prove that the result in (d) is correct.