IEOR 6711: Stochastic Models, I

Fall 2010, Professor Whitt, Final Exam

SOLUTIONS

There are four questions, each with several parts. Question 2 is longer than the others and thus counts more.

1. The Eight (8) Subway Line.

A new subway line has been added to the West Side for the convenience of Columbia students. It has six stations. Going north, it starts at 88^{th} street (station 1) and has stops at 98^{th} street (station 2), 108^{th} street (station 3), 118^{th} street (station 4), 128^{th} street (station 5) and 138^{th} street (station 6). It can change tracks and directions at the two end points, so that the trains travel in a loop, going north from station 1 to station 6 and then back south from station 6 to station 1, where it then goes north again. Subway trains follow a strict schedule: The travel time between successive stations is constant, equal to 2 minutes. There are two subway trains, one starting north from station 1 and the other starting south from station 6. Thus, at station 2, the intervals between successive trains in a specified direction are exactly 10 minutes. That is, a southbound train comes to station 2 every 10 minutes and also a northbound train comes to station 2 every 10 minutes. At the end stations, the story is different; e.g., at station 1 all arriving subway trains are southbound from 2, but one arrives every 10 minutes.

Customers arrive at station i to use the subway according to a Poisson process with rate λ_i per minute. Suppose that the subway has unlimited capacity and that the time to load and unload passengers can be ignored. Suppose that each customer entering station i gets off at station j with probability $P_{i,j}$, independently of all other customers (where $P_{i,i} = 0$). Suppose that people get on subways only in the direction they want to go.

(a) Give an expression for the expected number of customers to get on the subway (necessarily going north) at each visit to station 1.

Let $N_1(t)$ be the number of customers that arrive to station 1 in the time interval $[0, t]$. This is a Poisson process with rate λ_1. The times between successive subways at station 1 is 10 minutes. Each subway comes from station 2 and then heads back north to station 2. The expected number that get on a subway at each visit to station 1 is the expected number of arrivals over an interval of length 10 minutes. Hence the mean is

$$E[N_1(10)] = 10\lambda_1.$$

(b) Suppose that 8 customers get on the subway at station 1 (necessarily going north) at time t. What is the probability that exactly 3 of these customers had to wait more than 4 minutes before getting on the subway?

Given that 8 customers got on the subway at time t, there must have been 8 arrivals in the 10-minute interval $[t - 10, t]$. Given this number, the actual arrival times of the 8 customers are distributed as independent random variables, each uniformly distributed over the ten-minute interval. The probability that each customer had to wait more than 4 minutes is thus $6/10 = 0.6$. The probability that exactly 3 of these customers had to wait more than 4 minutes before getting on the subway is given by the binomial probability

$$b(3; 8, 0.6) = \frac{8!}{3!5!}(0.6)^3(0.4)^5 = 0.12386$$

(This is exploiting one of the basic properties of the Poisson process.)

(c) Give an expression for the probability that the number of customers getting off the northbound subway at a visit to station 4 is exactly j.

The number of people to get on the northbound subway at station at station i that get off at station 4 is an independent thinning (with probability $P_{i,4}$) of the number that gets on the northbound subway at station i, which has mean $10\lambda_i$, and is thus itself a Poisson random variable with mean $10\lambda_i P_{i,4}$. The numbers for different starting stations i are independent Poisson random variables, because the arrival processes at the different stations are independent Poisson processes. Finally, the sum of independent Poisson random variables is again a Poisson random variable with a mean equal to the sum of the component means. Let D_4 be the number of people to get off the northbound subway at station 4. The expected number of customers to get off is Thus,

$$P(D_4 = j) = \frac{e^{-m_4} m_4^j}{j!}$$

where

$$m_4 = E[D_4] = 10\lambda_1 P_{1,4} + 10\lambda_2 P_{2,4} + 10\lambda_3 P_{3,4}.$$

(d) Give an expression for the probability that, simultaneously, the number of customers getting off the northbound subway at a visit to station 4 is j and the number getting off at the next stop, at station 5, is k.

Independent thinnings of Poisson random variables become independent Poisson random variables. Thus these are independent random variables. Hence,

$$P(D_4 = j, D_5 = k) = \frac{e^{-m_4} m_4^j}{j!} \times \frac{e^{-m_5} m_5^j}{j!}$$

where m_4 is defined in part (e) and

$$m_5 = E[D_5] = 10\lambda_1 P_{1,5} + 10\lambda_2 P_{2,5} + 10\lambda_3 P_{3,5} + 10\lambda_4 P_{4,5}.$$

(e) Suppose that $\lambda_i = 2$ for all i and $P_{i,j} = 1/5$ for all j with $j \neq i$. How can you determine a convenient accurate approximation for the probability that the number of customers getting off the northbound subway at one specified visit to station 5 is greater than 20? Is that probability more than 1/20?

Applying part (d), we can insert these numbers to get

$$m_5 = E[D_5] = 10\lambda_1 P_{1,5} + 10\lambda_2 P_{2,5} + 10\lambda_3 P_{3,5} + 10\lambda_3 P_{3,5} = 4 \times (10 \times 2 \times \frac{1}{5}) = 16.$$

Hence, D_5 has a Poisson distribution with a mean of $m_5 = E[D_5] = 16$. We can now use a normal approximation for the Poisson distribution, which is appropriate when the mean is not too small.

$$\begin{aligned} P(D_5 > 20) &= P\left(\frac{D_5 - E[D_5]}{\sqrt{Var(D_5)}} > \frac{20 - E[D_5]}{\sqrt{Var(D_5)}}\right) \\ &\approx P\left(N(0,1) > \frac{20 - E[D_5]}{\sqrt{Var(D_5)}}\right) = P\left(N(0,1) > \frac{20-16}{\sqrt{16}}\right) \\ &= P(N(0,1) > 1) \approx 0.16 \end{aligned}$$

Even without a table, you should know that $P(N(0,1) > 1) > 0.05$.

For a slightly more refined approximation, we could account for the integer-valued random variable that we are dealing with. We would then look at

$$P(D_5 > 20) = P(D_5 > 20.5) \approx P(N(0,1) > (20.5 - 16)/4) = P(N(0,1) > 1.125) \approx 0.13,$$

which is still well above 0.05.

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2. **The Movement of a Taxi**

A continuously operating taxi serves three locations: A, B and C.

idle times:

The taxi sits idle at each location an exponential length of time before departing to make a trip to one of the other two locations. The mean idle times are 2 minutes at A, 1 minute at B and 2 minutes at C. The idle times and travel times are mutually independent.

transition probabilities:

From A, the taxi next goes to B with probability 1/3 and to C with probability 2/3.
From B, the taxi next goes to A with probability 1/2 and to C with probability 1/2.
From C, the taxi next goes to B with probability 1/3 and to A with probability 2/3.

travel times:

The travel times between A and B in either direction are uniformly distributed in the interval $[5, 15]$ minutes.
The travel times between A and C in either direction are uniformly distributed in the interval $[20, 60]$ minutes.
The travel times between B and C in either direction are uniformly distributed in the interval $[20, 40]$ minutes.

Please provide explicit numerical answers to all questions below, showing your method, unless indicated otherwise.

(a) What is the long-run proportion of all taxi trips starting from location A?

The transition matrix among locations, based on the transition probabilities given directly above, is

$$P = \begin{array}{c} A \\ B \\ C \end{array} \left(\begin{array}{ccc} 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \\ 2/3 & 1/3 & 0 \end{array} \right)$$

By symmetry, it is evident that the stationary vector must satisfy $\pi_A = \pi_C$. We thus easily solve $\pi = \pi P$ to get $\pi = (3/8, 2/8, 3/8)$. The long-run proportion of all trips starting from A is $\pi_A = 3/8$.

(b) What is the long-run proportion of time that the taxi's most recent stop was at location A?

We can solve this problem in (at least) two ways. First, in the context of the three states defined in part (a), we can compute the mean time spent in each state. We get the mean vector

$$m \equiv (m_A, m_B, m_C) = (32, 21, 116/3).$$

We then apply Theorem 4.8.3 of Ross to get the limiting probability. Let α_A be the long-run proportion of time that the last stop was in location A. (The taxi may be idle at A or traveling away from A, toward either B or C.) We then have by Theorem 4.8.3 of Ross that

$$\alpha_A = \frac{\pi_A m_A}{\pi_A m_A + \pi_B m_B + \pi_C m_C} = \frac{(96/8)}{(254/8)} = \frac{96}{254} = \frac{48}{127} \approx 0.378$$

The second way is to first increase the number of states to distinguish between being idle and traveling. That is done in part (c) below. After doing that, the answer becomes the long-run proportion of time the taxi is in one of the three states A, (A, B) and (A, C), where A means idle at A and (A, B) means traveling from A to B. From the details given below in part (c), we see that this long run proportion is

$$\frac{(96/16)}{(254/16)} = \frac{96}{254} = \frac{48}{127} \approx 0.378$$

Fortunately, the answers agree.

(c) What is the long-run proportion of time that the taxi is idle at location A?

Again, we can solve this in several ways. For this question, it is convenient to add six more states: (A, B), (A, C), (B, A), (B, C), (C, A) and (C, B), with (A, B) meaning that the taxi is traveling from A to B. We then have the 9-state transition matrix

$$P = \begin{array}{c} A \\ B \\ C \\ (A,B) \\ (A,C) \\ (B,A) \\ (B,C) \\ (C,A) \\ (C,B) \end{array} \begin{pmatrix} 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 1/3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From one of the three locations, we go to traveling; from traveling, we go next to one of the three locations. From the answer to part (a), it is easy to determine the stationary probability vector for this 9×9 chain. Since we are at the locations on half the transitions, we divide the previous probabilities by 2. We then get $\pi_{(A,B)} = \pi_A P_{A,B}$ using the 3×3 transition matrix on the right, and so forth. Hence, it is easy to see that the stationary probability vector for this 9×9 discrete-time Markov chain must be

$$\pi = (3/16, 2/16, 3/16, 1/16, 2/16, 1/16, 1/16, 2/16, 1/16).$$

This is also easily verified by direct calculation: We see that indeed $\pi = \pi P$ for this 9-state DTMC. (This representation plays a role in part (i) below, so I am anticipating that in my approach, but nevertheless this approach is natural.)

We are now ready to answer the question. Let α_A be the long-run proportion of time spent by the taxi idle at location A. We then have by Theorem 4.8.3 of Ross in this more detailed context that

$$\alpha_A = \frac{\pi_A m_A}{\sum_i \pi_i m_i},$$

where m_i is the mean time spent in state i. The mean vector is

$$m = (m_A, m_B, m_C, m_{(A,B)}, m_{(A,C)}, m_{(B,A)}, m_{(B,CB)}, m_{(C,A)}, m_{(C,B)}) = (2, 1, 2, 10, 40, 10, 30, 40, 30).$$

Hence,

$$\alpha_A = \frac{(3/16)2}{\sum_i \pi_i m_i} = \frac{(6/16)}{254/16} = \frac{6}{254} = \frac{3}{127} \approx 0.0236$$

On the other hand, we can exploit renewal theory directly. We can start by identifying an embedded renewal process. Let the times of successive arrivals to A constitute renewals. We can write

$$\alpha_A = \frac{m_A}{m_{A,A}} \quad \text{so that} m_{A,A} = \frac{m_A}{\alpha_A} = \frac{32}{96/254} = \frac{254}{3},$$

applying part (b). We then write

$$\frac{E[\text{reward per cycle}]}{E[\text{length of cycle}]} = \frac{2}{254/3} = \frac{6}{254} = \frac{3}{127} \approx 0.0236$$

(d) Let $P_t(A)$ be the probability that the taxi is idle at location A at time t. Does $P_t(A)$ converge to a proper limit as $t \to \infty$? Why or why not? If so, what is that limit?

The answer is YES by Proposition 4.8.1, which in turn is implied by the limit theorem for alternating renewal processes, Theorem 3.4.4, which in turn is implied by the key renewal theorem. The limit is the same as the answer in part (b). First, all the time random variables have densities, so there is no problem with non-lattice. The alternating renewal process step is useful, because the function $h(t)$ in the renewal equation that must be d.R.i. (directly Riemann integrable) is nondecreasing and bounded, which is a convenient condition for d.R.i. Specifically, the renewal equation is

$$g(t) = h(t) + \int_0^t g(t-s)\, dF(s),$$

and its solution is

$$g(t) = h(t) + \int_0^t h(t-s)\, dm(s),$$

where here $g(t) = P_t(A)$ and $h(t) = P(U > t)$, where U is the length of the "up" or "on" interval in the alternating renewal process with cycle cdf F and $m(t)$ is the renewal function associated with F.

(e) What is the rate at which the taxi makes trips departing from location A heading toward location B?

First, we might need to clarify that rate means per unit of time. The long-run rate that the taxi makes trips from A is $1/m_{A,A}$, where $m_{A,A}$ is the mean time between arrivals to A. The instants between arrivals to A can serve as an embedded renewal process. A proportion $P_{A,B}$ of these visits are followed by a trip to B. So the long-run rate of trips from A to B is

$$P_{A,B}/m_{A,A}.$$

On the other hand, we know that $\alpha_A = m_A/m_{A,A}$ by Proposition 4.8.1. Hence,

$$\frac{P_{A,B}}{m_{A,A}} = \frac{\alpha_A P_{A,B}}{m_A} = \frac{(3/127)(1/3)}{2} = \frac{1}{254}.$$

(f) What is the long-run conditional probability that the taxi will come next to location B, given that the taxi is now traveling away from location A?

$$\lim_{t\to\infty} P(X(t) = (A,B) | X(t) \in \{(A,B) \cup (A,C)\}) = \frac{\lim_{t\to\infty} P(X(t) = (A,B))}{\lim_{t\to\infty} P(X(t) \in \{(A,B) \cup (A,C)\})}$$
$$= \frac{\pi_{(A,B)} m_{(A,B)}}{\pi_{(A,B)} m_{(A,B)} + \pi_{(A,C)} m_{(A,C)}} = \frac{(1/16)10}{(1/16)10 + (2/16)40} = \frac{1}{9}.$$

Note that this conditional probability does not simply equal $P_{A,B} = 1/3$. We need to take time into account.

(g) What is the long-run proportion of time that the taxi is traveling from A to C and the remaining time before getting to C is at least 30 minutes?

This is a variant of Theorem 4.8.4 in Ross. Let $m_{(A,C),(A,C)}$ be the mean time between beginning a trip from A to C. Let $T_{A,C}$ be the uniformly distributed travel time on a trip from A to C. let $m_{A,C} \equiv E[T_{A,C}]$. We already have two expressions for $\alpha_{(A,C)}$, one being $m_{A,C}/m_{(A,C),(A,C)}$ and the other from the reasoning in part (c), involving Theorem 4.8.3 of Ross.

$$\begin{aligned}
\lim_{t\to\infty} P(X(t) = (A,C), Y(t) > 30) &= \frac{E[(T_{A,C} - 30)^+]}{m_{(A,C),(A,C)}} \\
&= \frac{\int_0^{\infty} P((T_{A,C} - 30)^+ > y)\, dy}{m_{(A,C),(A,C)}} \\
&= \frac{\int_0^{30} P(U(20,60) - 30 > y\, dy}{m_{(A,C),(A,C)}} \\
&= \frac{\int_0^{30} (30 - y)/40\, dy}{m_{(A,C),(A,C)}} \\
&= \frac{(900)/80}{m_{(A,C),(A,C)}} = \frac{(900)/80)\alpha_{(A,C)}}{m_{(A,C)}} \\
&= \left(\frac{900}{80}\right)\left(\frac{(80/254)}{40}\right) = \left(\frac{45}{4}\right)\left(\frac{1}{127}\right) = \frac{45}{508} \approx 0.089
\end{aligned}$$

(h) Which of the previous answers would change if the travel times were changed from uniform to exponential with the same mean? (You need not do any new computations?)

Only the previous part, part (g), would have a different answer. All the others had formulas that depend only on the mean travel time.

(i) Suppose that the travel times are indeed changed from uniform to exponential with the same mean. Let $X(t)$ be the state of the taxi at time t, e.g., idle at A or traveling from A to B. Give an explicit formula (not numerical value) for the conditional probability

$$P(X(2) = \text{ idle at } B \quad \text{and} \quad X(7) = \text{ idle at } C | X(0) = \text{ idle at } A).$$

Under the new exponential assumption, the SMP using the 9 states becomes a CTMC. With the state notation in the solution to part (c), we have

$$\begin{aligned}
&P(X(2) = \text{ idle at } B \quad \text{and} \quad X(7) = \text{ idle at } C | X(0) = \text{ idle at } A) \\
&\equiv P(X(2) = B, X(7) = C | X(0) = A) = P_{A,B}(2) P_{B,C}(5),
\end{aligned}$$

where $P_{i,j}(t) \equiv P(X(t+s) = j | X(s) = i)$ is the transition probability for the CTMC. The transition probability $P_{i,j}(t)$ in turn is an element of the transition matrix $P(t)$, which can be obtained by solving one of the ODE's $\dot{P}(t) = QP(t)$ or $\dot{P}(t) = P(t)Q$. More directly, the solution of these ODE's can be represented explicitly as the matrix exponential

$$P(t) = e^{Qt} \equiv \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!},$$

where Q is the 9×9 rate matrix for the CTMC. Here the rate matrix is

$$Q = \begin{array}{c} A \\ B \\ C \\ (A,B) \\ (A,C) \\ (B,A) \\ (B,C) \\ (C,A) \\ (C,B) \end{array} \left(\begin{array}{ccccccccc} -1/2 & 0 & 0 & 1/6 & 2/6 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 2/6 & 1/6 \\ 0 & 1/10 & 0 & -1/10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/40 & 0 & -1/40 & 0 & 0 & 0 & 0 \\ 1/10 & 0 & 0 & 0 & 0 & -1/10 & 0 & 0 & 0 \\ 0 & 0 & 1/30 & 0 & 0 & 0 & -1/30 & 0 & 0 \\ 1/40 & 0 & 0 & 0 & 0 & 0 & 0 & -1/40 & 0 \\ 0 & 1/30 & 0 & 0 & 0 & 0 & 0 & 0 & -1/30 \end{array} \right)$$

We get these entries as follows. For the transition rate from A to (A,B), we multiply the reciprocal of the mean idle time by the transition probability, getting $1/2 \times 1/3 = 1/6$. For the transition rate from (A,B) to B, we use the reciprocal of the mean travel time. The diagonal entries are minus the off-diagonal row sum.

Alternatively in this framework you could use a uniformization expression in the 9-state framework above. It is also possible to give a more complicated SMP expression, without introducing the extra states, but that is rather cumbersome.

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3. Uniform random numbers

Consider a sequence of i.i.d. uniform random numbers $\{U_n : n \geq 1\}$, where U_n is uniformly distributed on the interval $[0,1]$. Let S_n be the sum of the first n uniform numbers, i.e.,

$$S_n \equiv U_1 + U_2 + \cdots + U_n, \quad n \geq 1,$$

with $S_0 \equiv 0$. Let F_n be the *fractional part* of S_n, defined by

$$F_n \equiv S_n - \lfloor S_n \rfloor, \quad n \geq 1,$$

where $\lfloor x \rfloor$ is the floor function, yielding the greatest integer less than or equal to the real number x. Let R_n be the *remainder* beyond n of the first partial sum to exceed n. That is let Z_n be the least integer k such that $S_k > n$, and let

$$R_n \equiv S_{Z_n} - n, \quad n \geq 1.$$

Let $\Rightarrow$ denote convergence in distribution.

(a) Is the stochastic process $\{F_n : n \ge 1\}$ a Markov process? Why or why not?

Yes, it is a Marlov process. The conditional probability distribution of F_{n+1} given that $F_n = x$, $0 \le x < 1$, and given F_k, $1 \le k < n$, depends only on $F_n = x$. In particular,

$$F_{n+1} = x + y \quad \text{if} \quad F_n = x \quad \text{and} \quad U_{n+1} = y \,\text{with}\, x + y < 1,$$

while

$$F_{n+1} = x + y - 1 \quad \text{if} \quad F_n = x \quad \text{and} \quad U_{n+1} = y \,\text{with}\, x + y \ge 1,$$

where U_{n+1} is independent of $U_1, \ldots U_n$ and thus also of $F_1, \ldots F_n$.

(b) Is the stochastic process $\{R_n : n \ge 1\}$ a Markov process? Why or why not?

Yes, it is a Markov process. The conditional probability distribution of R_{n+1} given that $R_n = x$, $0 \le x < 1$, and given R_k, $1 \le k < n$, depends only on $R_n = x$. In particular,

$$R_{n+1} = x + y - 1 \quad \text{if} \quad R_n = x \quad \text{and} \quad U_{Z_{n+1}} = y \text{ with}\, x + y > 1,$$

where $U_{Z_{n+1}}$ is independent of $U_1, \ldots U_{Z_n}$ and thus also of $R_1, \ldots R_n$,

$$R_{n+1} = x + y_1 + y_2 - 1 \quad \text{if} \quad R_n = x,\, U_{Z_{n+1}} = y_1 \quad \text{and} \quad U_{Z_{n+2}} = y_2 \,\text{with}\, x + y_1 \le 1 < x + y_1 + y_2,$$

where $U_{Z_{n+i}}$ is independent of $U_1, \ldots U_{Z_n}$ and thus also of $R_1, \ldots R_n$, and, more generally,

$$R_{n+1} = x + \sum_{i=1}^{k} y_i - 1 \quad \text{if} \quad R_n = x,\, U_{Z_{n+i}} = y_i,\, 1 \le i \le k \,\text{with}\, x + \sum_{i=1}^{k-1} y_i \le 1 < x + \sum_{i=1}^{k} y_i.$$

where U_{Zn+i} is independent of $U_1, \ldots U_{Z_n}$ and thus also of $R_1, \ldots R_n$,

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(c) Prove that there exists a random variable F such that $F_n \Rightarrow F$ as $n \to \infty$ and determine the probability distribution of the random variable F.

The stochastic process $\{F_n : n \ge 1\}$ is a discrete-time Markov process, but the state space is uncountably infinite, so that it falls outside of the scope of the material we have covered. We must instead just think about what is going on. First, we directly have $F_1 = S_1 \equiv U_1$ uniformly distributed and, given that F_n is uniformly distributed, we can easily prove that the distribution of F_{n+1} is again uniformly distributed on $[0, 1]$. Hence, F_n has a uniform distribution on $[0, 1]$ for all n. Hence, we trivially have $F_n \Rightarrow F$, where F is a random variable uniformly distributed on $[0, 1]$; i.e.,

$$P(F \le x) = x, \quad 0 \le x \le 1.$$

(d) Prove that there exists a random variable R such that $R_n \Rightarrow R$ as $n \to \infty$ and determine the probability distribution of the random variable R.

This is the main part of the question. We should use renewal theory here. The idea is to recognize that this is a simple and direct application of renewal theory, even if somewhat in a disguised setting. We are concerned with the residual lifetime at time $t = n$. The discrete time framework is a red herring; in this part it plays no role. However, the problem emphasizes the difference between the two stochastic processes involved.

Notice that the stochastic process $\{R_n : n \ge 1\}$ evolves differently than the stochastic process $\{F_n : n \ge 1\}$. In fact, R_n is the standard excess process in renewal theory, i.e.,

$$R_n = Y(n), \quad \text{where} \quad Y(t) \equiv S_{N(t)+1} - t, \quad t \ge 0.$$

We already know that $Y(t) \Rightarrow Y$ as $t \to \infty$, so it also necessarily does through any subsequence of time arguments that go to infinity. Thus, $Y(n) \Rightarrow Y$ as $n \to \infty$ too. If we look at the stochastic process $\{1_{Y(t) \ge x} : t \ge 0\}$ for any fixed x, we see that it corresponds to an alternating renewal process with an initial up time where the process assumes the value 0 and then a down time, where it assumes the value 0. Hence, the limit follows from the limit for alternating renewal processes, Theorem 3.4.4 of Ross, which in turn follows from the key renewal theorem. Since the distribution of the time between renewals is uniform, it is non-lattice. The random variable R thus has the *stationary excess distribution* associated with the uniform distribution, i.e.,

$$P(R \le x) = \frac{1}{E[U]} \int_0^x P(U > y)\, dy = 2 \int_0^x 1 - y\, dy = 2x - x^2, \quad 0 \le x \le 1.$$

(e) Compare the probability distributions of the random variables F and R. Are the distributions the same? Are the distributions stochastically ordered? Or do neither of these relations hold?

First, we see that the distributions of F and R are different. The distribution of F is uniform, while the distribution of R is the distribution of U_e, having the stationary-excess distribution of a uniform distribution. It is easy to see that these random variables are stochastically ordered, i.e.

$$R \le_{st} F \quad \text{or} \quad P(R \le x) \ge P(F \le x) \quad \text{for all} \quad x,$$

because

$$P(R \le x) = (2x - x^2) \ge x = P(F \le x) \quad \text{for all} \quad x, \quad 0 \le x \le 1.$$

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4. New Airport Security Check (30 points)

A new elaborate airport security check has been designed with three inspection stations. At each inspection station, passengers are processed one at a time in order of arrival at that station. There is ample waiting space at each station. Suppose that the processing times at the stations are exponentially distributed random variables. Let the mean processing times be 10 seconds at station 1, 20 second at station 2 and 10 minutes at station 3 (more serious inspection).

Suppose that passengers may enter the security check system at either station 1 or station 2. Suppose that passengers arrive at station 1 according to a Poisson process with rate 2 per minute; suppose that passengers arrive at station 2 according to a Poisson process with rate 1 per minute.

Suppose that 1/4 of all passengers undergoing inspection at station 1 must repeat inspection at station 1, where they are required to go to the end of the queue at station 1. Suppose that 1/2 of all passengers undergoing inspection at station 1 must go next to inspection at station 2, where they are required to go to the end of the queue at station 2. Suppose that 1/100 of all passengers completing inspection at station 1 must go next to inspection at station 3. Suppose that 1/2 of all passengers undergoing inspection at station 2 must go next to inspection at station 1, where they are required to go to the end of the queue at station 1. Suppose that no customers completing inspection at station 2 need to immediately repeat inspection at station 2. Suppose that 1/50 of all passengers completing inspection at station 2 must go next to inspection at station 3. The remaining passengers completing inspection at stations 1 and 2 leave the system. All passengers completing inspection at station 3 leave the system after completing inspection at station 3. Suppose that 1/1000 passengers completing inspection at station 3 are classified as a serious security risk.

(a) Specify the customary (required) assumptions on the model elements that make the stochastic process recording the number of passengers at each of the three stations a continuous-time Markov chain (CTMC), and specify the model. Henceforth assume that these assumptions are in force.

Here we have an open Jackson Markovian network of single-server queues. You might want to look at the journal *Production and Operations Management*, vol. 17, No. 6, November-December 2008, p. i.

There are **two critical assumptions** that need to be made:

(i) First, we need to assume that the two Poisson processes and all the service times are mutually independent. That means that all service times are mutually independent, the two Poisson processes are independent of each other, and the service times are independent of the arrival processes.

(ii) Second, we need to assume that we have Markovian routing. We need to assume that the routing of each passenger after completing processing at any station occurs with probability equal to the specified proportion, and is independent of the system history up to that time.

Given those assumptions, **the model is specified by:** (i) the external arrival rates at each station, given by the vector

$$(\lambda_{e,1}, \lambda_{e,2}, \lambda_{e,3}) = (2, 1, 0) \quad \text{per minute,}$$

(ii) the vector of number of servers at each station and service rates (per server),

$$(s_1, s_2, s_3) = (1, 1, 1) \quad \text{and} \quad (\mu_1, \mu_2, \mu_3) = (6, 3, 0.1) \quad \text{per minute,}$$

and (iii) the Markovian routing matrix

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1/4 & 1/2 & 1/100 \\ 1/2 & 0 & 1/50 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Given this model, what is the long-run proportion of time that station 2 is busy? What is the long-run proportion of time that station 1 is busy? What is the long-run average waiting time per passenger to complete the entire inspection process.

In order to determine the long-run performance, we need to first solve for the overall "net" arrival rates at each station, accounting for the internal flows as well as the external flows. In other words, we need to solve the **traffic rate equations:**

$$\Lambda = \Lambda_e(I - P)^{-1}.$$

However, we can initially ignore station 3 because there is no flow from station 3 back to stations 1 and 2. Hence, it suffices to solve two equations in the two unknowns λ_1 and λ_2:

$$\begin{aligned}\lambda_1 &= 2 + \frac{\lambda_1}{4} + \frac{\lambda_2}{2} \\ \lambda_2 &= 1 + \frac{\lambda_1}{2}.\end{aligned}$$

We easily solve these two equations to get

$$(\lambda_1, \lambda_2) = (5.0, 3.5) \quad \text{per minute.}$$

We next investigate stability by computing the **traffic intensity at each station**. We get the traffic intensities

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{5}{6} < 1 \quad \text{and} \quad \rho_2 = \frac{\lambda_2}{\mu_2} = \frac{3.5}{3} = \frac{7}{6} > 1.$$

Since $\rho_2 > 1$, we see that the system is **unstable**. The queue at station 2 will explode, diverging to ∞ as $t \to \infty$. Station 2 will always be busy in the long run. Thus the CTMC fails to converge to a proper steady-state distribution.

Nevertheless, we can continue to see what would happen at station 1. The processing rate at station 2 will be the maximum possible processing rate of 3 per minute. Half that output will go back to station 1. Hence, we can determine a revised traffic rate equation for station 1:

$$\begin{aligned}\lambda_1 &= 2 + \frac{\lambda_1}{4} + \frac{\lambda_2}{2} \\ &= 2 + \frac{\lambda_1}{4} + \frac{3}{2}\end{aligned}$$

so that

$$\frac{3\lambda_1}{4} = \frac{7}{2} \quad \text{and} \quad \lambda_1 = \frac{14}{3}.$$

So, after reducing the flow into 1 from station 2 to 3/2, we get a net flow rate into station 1 of 14/3. The resulting traffic intensity at station 1 is

$$\rho_1 = \frac{14/3}{6} = \frac{7}{9} < 1.$$

We first conclude that the model is not stable. A proper steady state does not exist. The long-run proportion of time that station 2 is busy is 1, while long-run proportion of time that station 1 is busy is 7/9.

Finally, we are asked about the long-run average waiting time. The long-run average is infinite, because a positive proportion will never be served. Passengers leave stations 1 and 2 from station 2 at rate $3 \times 1/2 = 1.5$; passengers leave stations 1 and 2 from station 1 at rate $14/3 \times 1/4 = 7/6$. The total rate in is $2 + 1 = 3$, while the total rate out is $1.500 + 1.167 = 2.67 < 3.00$. So the total population in the system is growing at rate 0.33 per minute in the long run.

(c) How do long-run proportions of time that stations 1 and 2 are busy in part (b) change when the arrival rate of passengers at station 1 from outside the system is changed from 2 per minute to 1 per minute? Henceforth (for all the remaining questions), assume that the arrival rate at station 1 from outside the system is indeed 1 per minute.

When we solve the new traffic rate equations in the two unknowns λ_1 and λ_2,

$$\begin{aligned} \lambda_1 &= 1 + \frac{\lambda_1}{4} + \frac{\lambda_2}{2} \\ \lambda_2 &= 1 + \frac{\lambda_1}{2}. \end{aligned}$$

we get

$$(\lambda_1, \lambda_2) = (3.0, 2.5) \quad \text{per minute.}$$

The associated traffic intensities are

$$(\rho_1, \rho_2) = (3.0/6.0, 2.5/3.0) = (1/2, 5/6).$$

It now remains to consider station 3. The arrival rate at station 3 is

$$\lambda_3 = \lambda_1 \times \frac{1}{100} + \lambda_2 \times \frac{1}{50} = 0.08 \quad \text{per minute} \tag{1}$$

The mean service time at station 3 is 10 minutes. Hence, the traffic intensity at station 3 is

$$\rho_3 = \frac{\lambda_3}{\mu_3} = (0.08)(10) = 0.8 < 1.$$

Hence all three traffic intensities are less than 1.

Now the system is stable. Now there exists a proper steady-state distribution. Let $Q_i(t)$ be the number of passengers at station i at time t, either in service or waiting. By Theorem 6.9 in the CTMC notes, the long-run steady-state distribution is the product of three geometric distributions

$$P(Q_1(t) = j_1, Q_2(t) = j_2, Q_3(t) = j_3) = \prod_{i=1}^{i=3} (1 - \rho_i)\rho_i^{j_i}. \tag{2}$$

The long-run proportion of time that station i is busy is ρ_i, where the vector of traffic intensities is

$$(\rho_1, \rho_2, \rho_3) = (1/2, 5/6, 4/5).$$

(d) Given the adjusted model in part (c), what is the probability that there is simultaneously 1 passenger at station 1, 2 passengers at station 2 and 3 passengers at station 3 at some time t after the system has been operating for a long time?

As in the solution to part (c), by Theorem 6.9 of the CTMC notes, we have equation (2). Hence,

$$P(Q_1(t) = 1, Q_2(t) = 2, Q_3(t) = 3) = (1-(1/2))(1/2)^1(1-(5/6))(5/6)^2(1-(4/5)(4/5)^3 = \frac{2}{675}.$$

(e) Given the model, what is the long-run proportion of all arriving passengers that will be classified as a serious security risk?

By equation (1) above, the arrival rate to station 3 is 0.08 per minute. The total external arrival rate is $\lambda_{e,1} + \lambda_{e,2} = 1 + 1 = 2$. Hence, a proportion 0.04 of all arrivals go to station 3. Then 1/1000 of these are deemed a serious security risk. That means the overall long-run proportion will be

$$0.04 \times 0.001 = 0.00004 = \frac{4}{100,000}$$

That is 0.004% of all passengers Of course, that does *not* tell us what proportion of passengers should be judged to be a serious security risk, because the effectiveness of the inspection is not included in the model. This is only how many passengers will be deemed to be a security risk. There are two possible errors: First, passengers who should be considered a risk may not be identified. Second, some of these passengers identified as potential risks may not actually present a problem.

(f) If possible, construct the reverse-time Markov chain associated with the CTMC specified in part (a), as adjusted in part (c).

The reverse-time chain requires that we start the system in equilibrium with the steady-state distribution, which has been determined in equation (2) above. When we do so, the reverse-time chain is also a CTMC, indeed also an open Markovian queueing network, like the forward process. We define reverse-time external arrival rates by

$$\overleftarrow{\lambda}_{e,i} = \lambda_i(1 - \sum_{j=1}^{3} P_{1,j}),$$

as in (6.18) of the CTMC notes. We let $\overleftarrow{\mu}_i = \mu_i$ for all i. We make $\overleftarrow{\lambda}_i = \lambda_i$ for all i, as in (6.17) of the CTMC notes. Finally, we let

$$\overleftarrow{P}_{j,i} = \frac{\lambda_i P_{i,j}}{\overleftarrow{\lambda}_j}$$

for all i and j, as in (6.20) of the CTMC notes.

We can continue and fill in the numerical details. First, we have the net flow rates

$$\overleftarrow{\lambda} = \lambda = (3.00, 2.50, 0.08)$$

Then we have the external arrival rates for the reverse-time chain

$$\overleftarrow{\lambda}_e = (18/25, 5/4, 2/25).$$

Next we have the reverse-time Markovian routing matrix

$$\overleftarrow{P} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 1/4 & 5/12 & 0 \\ 3/5 & 0 & 0 \\ 3/8 & 5/8 & 0 \end{pmatrix}$$

You can then check that the net flow rates satisfy the reverse-time traffic-rate equations

$$\overleftarrow{\lambda}_j = \overleftarrow{\lambda}_{e,j} + \sum_{i=1}^{i=3} \overleftarrow{\lambda}_i \overleftarrow{P}_{i,j}$$

for each j, $1 \le j \le 3$.

(g) Identify conditions, if possible, under which the stochastic process recording the number of passengers at each of the three stations is a time-reversible continuous-time Markov chain.

The process is not time-reversible, because of the direction of flow. It is *not* possible. Each station alone would be an $M/M/1$ queue, and the stochastic process $Q_i(t) : t \ge 0\}$ would be a birth-and-death process, and so reversible, but the three-dimensional process cannot be reversible with the given routing matrix P.

(h) Indicate how to efficiently prove that the result in (d) is correct.

We apply Theorem 6.8 of the CTMC notes. We need to verify conditions (6.9) and (6.10) there. That needs to be done, just as in the alternate proof of Theorem 6.7 in the CTMC notes. We do not write all this out here, but ideally the two conditions are specified and then, even better, is to actually show that these conditions are satisfied.
