IEOR 6711: Stochastic Models, I

Fall 2012, Professor Whitt, Final Exam

There are four questions, each with several parts.

1. Customers Coming to an Automatic Teller Machine (ATM) (30 **points)**

Customers arrive one at a time at a single automatic teller machine (ATM), with unlimited waiting space, to withdraw money. Customers **arrive** according to a Poisson process with rate $\lambda \equiv 48$ per hour. The **service times** of successive customers at the ATM can be regarded as i.i.d. random variables, distributed as the random variable S having a gamma distribution with mean $ES \equiv 1$ minute and standard deviation $\sqrt{Var(S)} \equiv 0.5$, and thus probability density function (pdf)

$$g(t) \equiv g_S(t) \equiv \frac{128t^3e^{-4t}}{3}, \quad t \geq 0,$$

and Laplace transform

$$\hat{g}(s) \equiv E[^{-sS}] \equiv \int_0^\infty e^{-st} g(t)\, dt = \left(\frac{4}{4+s}\right)^4 .$$

The successive **withdrawals** can be regarded as i.i.d. random variables distributed as W having a gamma distribution with mean $EW \equiv 100$ dollars and standard deviation $SD(W) \equiv 110$ and thus Laplace transform

$$E[e^{-sW}] \equiv \left(\frac{1}{1+121s}\right)^{(1/1.21)} .$$

[first part: arrivals (numerical answers plus explanation desired)] (4 **points)**

(a) Suppose that 20 arrivals come during a given hour. What are the mean and variance of the number of these arrivals that come during the first 15 minutes of that hour?

(b) What is the probability that the first arrival completes service before the second customer arrives? (Assume that the system is initially empty.)

[second part: total money withdrawn (numerical answers plus explanation desired)] (6 **points)**

(c) What are the mean and variance of the total amount of money withdrawn by all customers during one hour?

(d) What is the expected conditional total amount withdrawn in a given hour, given that the amount withdrawn in the previous hour is exactly two times the mean?

(e) What is the approximate probability (to within 0.1) that the total amount of money withdrawn by all customers during one hour exceeds \$10,000?

[third part: number of customers at the ATM] (20 **points)**

Let $X(t)$ be the number of customers at the ATM at time t.

(f) Is $\{X(t) : t \geq 0\}$ an irreducible aperiodic continuous-time Markov chain? Explain.

(g) Identify random times T_n, $n \geq 0$, such that $\{X(T_n) : n \geq 0\}$ is an irreducible aperiodic discrete-time Markov chain with state space $\{0, 1, 2, \ldots\}$.

(h) For the DTMC identified in part (g), exhibit the transition probabilities.

(i) For the DTMC identified in parts (g) and (h), prove that $X(T_n) \Rightarrow L$ as $n \rightarrow \infty$ for some random variable L with $P(L < \infty) = 1$ and characterize the probability distribution of L.

(j) State and prove a limit theorem describing the distribution of $L(\lambda)$ (regarded as a function of λ) if we let the arrival rate λ increase from 48 per hour to 60 per hour (while holding the service-time distribution unchanged).

2. The Department of Motor Vehicles (DMV) (26 points)

You can get a new automobile driver's license at the New York Department of Motor Vehicles (DMV) on 34th Street. The standard process for getting a new license involves passing through three stages of service. First, you get in a single line to wait for your turn to be served by a single clerk to get the correct form; second, you get in a single line to wait for a photographer to take your picture; and, third, you wait in a single line (you actually sit in a waiting room with order maintained by being assigned a number) for one of several clerks to complete the processing, prepare your license and collect your money. Your goal is to analyze the performance in this service system.

Make the following assumptions: Suppose that customers arrive according to a Poisson process with a rate of 2 per minute. Suppose that the service times at each step have exponential distributions. Suppose that the service times of different customers and of the same customers at different steps are all mutually independent. Suppose that all arriving customers go to the first clerk. Suppose the mean service time there is 15 seconds. Suppose that there is unlimited waiting space and that all customers are willing to wait until they can be served (Nobody gets impatient and abandons.) Suppose that only 1/4 of all arriving customers seek new licenses and must complete the second and third steps. The other 3/4 of the arriving customers go elsewhere after finishing the first stage. Suppose that the mean time required for the photographer to take a picture is 1 minute. Suppose that the mean time for each of the final clerks to complete the processing, prepare your license and collect your money is 10 minutes.

[first part: basic performance analysis] (10 points)

(a) How many clerks are need at the third stage (to complete the processing, prepare your license and collect your money) in order for the total customer arrival rate at the third stage to be strictly less than the maximum possible service rate (assuming all servers are working)?

Henceforth assume that there are 8 clerks at the third stage.

(b) Suppose that the first customer of the day (who finds a completely empty system upon arrival) wants to get a new driver's license. What are the mean and variance of the total time that this initial customer must spend at the DMV in order to get the license?

(c) Suppose that the second customer of the day also wants to get a new driver's license, and suppose that this second customer finds the first customer still being served by the first clerk when he arrives. What is the expected total time that this second customer must spend at the DMV in order to get the license?

(d) Give an expression for the steady-state probability that there are 3 customers either waiting or being served at the first clerk, and 5 customers either waiting or being served at the photographer.

(e) Give an expression for the steady-state probability that exactly 4 customers complete service from the photographer during one specified minute in steady state.

[second part: supporting theory; carefully state theorems applied.] (10 **points)**

Let $X_j(t)$ be the number of customers at station j, either waiting or being served.

(f) Prove or disprove: With an appropriate stationary distribution, the stochastic process $\{(X_1(t), X_2(t), X_3(t)) : t \geq 0\}$ is a time reversible irreducible CTMC.

(g) Prove or disprove: The stochastic process $\{(X_1(t), X_2(t), X_3(t)) : t \geq 0\}$ has a proper limiting distribution, i.e.,

$$\lim_{t\to\infty} P(X_1(t) = j_1, X_2(t) = j_2, X_3(t) = j_3)|X_1(0) = i_1, X_2(0) = i_2, X_3(0) = i_3) = \alpha_{(j_1,j_2,j_3)}$$

for all vectors of nonnegative integers (j_1, j_2, j_3) and (i_1, i_2, i_3), where

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} \alpha_{(j_1,j_2,j_3)} = 1.$$

[third part: starting over] (6 **points)**

Suppose that, with probability 1/5, independent of the history, each customer after completing all three stages of service has to return immediately to the end of the first queue and start the process over, with new independent service times.

(h) What is the new answer to part (a) above under this new condition?

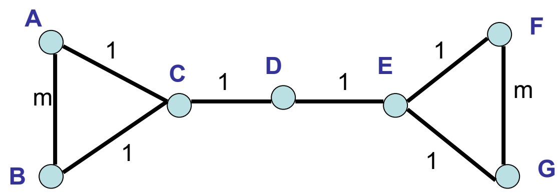
(i) How do the answers to parts (f) and (g) above change under this new condition?

(j) Prove or disprove: Under this new condition, the arrival process at station 2 when the system is in steady state (with a stationary initial distribution) is a Poisson process.

3. Random Walk on a Graph (24 points)

Consider the graph shown in the figure on the top of the next page. There are 7 nodes, labeled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Six of the arcs have weight 1, while two of the arcs have weight m. Consider a random walk on this graph, which moves randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc, independent of the history prior to reaching the current node. Thus the probability of moving from node A to node C in one step is $1/(1+m)$, while the probability of moving from node C to node A in one step is $1/(1+1+1)$ $= 1/3$. Let X_n be the node occupied after the n^{th} step of the random walk. Suppose that $X_0 = A$.

Random Walk on a Graph



(a) (4 points) For any nodes A and B, let $T_{A,B}$ be the first passage time (number of steps) required to go from A to B, with $T_{A,A} \geq 1$. Calculate $E[T_{A,A}]$. Briefly explain.

(b) (4 points) Let $Z_{A,B,C}$ be the number of visits to B starting in A before hitting C. Give an expression for $E[Z_{A,B,F}]$ and justify your answer.

(c) (3 points) Calculate numerical values for $E[Z_{A,B,C}]$ and $E[Z_{F,G,E}]$.

(d) (4 points) Calculate $E[z^{Z_{A,B,C}}]$.

(e) (3 points) Prove or disprove: If a discrete-time stochastic process $\{X_n : n \geq 0\}$ has a unique proper limiting distribution as $n \to \infty$, then that limiting distribution is the unique stationary distribution.

(f) (3 points) Prove or disprove: Every finite-state DTMC $\{X_n : n \geq 0\}$ has a stationary distribution.

(g) (3 points) For the random walk on the graph, prove that there exists a DTMC with transition matrix denoted by $P(\infty)$ such that

$$P(m) \to P(\infty) \quad \text{as} \quad m \to \infty,$$

where by convergence of matrices we mean that all elements converge. Exhibit all stationary probability vectors for the DTMC with transition matrix $P(\infty)$.

4. Sums of i.i.d Positive Random Variables (20 points)

Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. positive random variables with pdf f and mean $E[X_n] = 2$. Let $S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$, and $S_0 \equiv 0$. For any $t > 0$, let

$$w(t) \equiv \sum_{n=1}^{\infty} P(S_n \leq t) \quad \text{and} \quad V(t) \equiv S_n - t,$$

for the $n \equiv n(t)$ such that $S_{n-1} \leq t < S_n$.

(a) Prove or disprove: $w(t) < \infty$ for all t, $0 \leq t < \infty$.

(b) Prove or disprove: $\lim_{t\to\infty} (w(t)/t) = c$ for some constant c, $0 < c < \infty$.

(c) Determine $E[e^{-sV(t)}]$.

(d) Prove or disprove: $V(t)$ converges in distribution to a proper limit as $t \to \infty$. If the limit exists, then identify it.