IEOR 6711: Stochastic Models, I

Fall 2013, Professor Whitt, Final Exam

There are three questions, each with several parts.

1. Customers Coming to a Group of Automatic Teller Machines (35 points)

Customers arrive one at a time to a **group of 4 ATM's** (automatic teller machines) to withdraw money. Customers **arrive** according to a Poisson process with rate $\lambda \equiv 1$ per minute. The **service times** of successive customers at the ATM can be regarded as i.i.d. random variables, each distributed as the random variable S, with mean $E[S] = 2$. However, the customers are **highly impatient** and are unwilling to wait if the 4 ATM's are all busy, so that they leave immediately if all ATM's are busy.

First Part: Exponential Service Times

For this first part assume that the service times are exponentially distributed.

(a) Identify an appropriate Markov process that can be used to analyze the steady-state and transient behavior of this system.

(b) What is the long-run proportion of time that all 4 ATM's are simultaneously busy?

(c) True or False: The stationary departure process of served customers is a Poisson process. Justify your answer.

Second Part: Uniform Service Times

For this second part, assume that the service times are uniformly distributed on the interval $[0, 4]$.

(d) What is the long-run proportion of time that all 4 ATM's are simultaneously busy?

(e) Identify an appropriate Markov process that can be used to justify your answer in part (d) and study the transient behavior of this system?

(f) Exhibit the steady-state distribution of the Markov process in part (e). (Be as explicit as possible.)

(g) Prove that the steady-state distribution in part (f) is valid, stating all theorems used, possibly including theorems not actually proved in the book or in class.

(h) Let T be the time after an arrival finds the system full in steady state until an ATM first becomes free. What is $P(T > 2 \quad \text{minutes})$? (explicit numerical answer desired, recall that $E[S] = 2$)

(i) Show that the stochastic process representing the number of busy ATM's at time t is a regenerative process by identifying the regeneration times.

(j) What is the expected time between successive regeneration times in part (i)?

2. A Taxi Stochastic Process (35 points)

A continuously operating taxi serves three locations: A, B and C.

idle times:

The taxi sits idle at each location an exponential length of time before departing to make a trip to one of the other two locations. The mean idle times are 2 minutes at A, 1 minute at B and 2 minutes at C. The idle times and travel times are mutually independent.

transition probabilities:

From A, the taxi next goes to B with probability $1/3$ and to C with probability $2/3$.
From B, the taxi next goes to A with probability $1/2$ and to C with probability $1/2$.
From C, the taxi next goes to B with probability $1/3$ and to A with probability $2/3$.

travel times:

The travel times between A and B in either direction are uniformly distributed in the interval $[5, 15]$ minutes.
The travel times between A and C in either direction are uniformly distributed in the interval $[20, 60]$ minutes.
The travel times between B and C in either direction are uniformly distributed in the interval $[20, 40]$ minutes.

(a) What is the long-run proportion of all taxi trips starting from location A?

(b) What is the long-run proportion of time that the taxi's most recent stop was at location A?

(c) What is the long-run proportion of time that the taxi is idle at location A?

(d) Let $P_t(A)$ be the probability that the taxi is idle at location A at time t. Does $P_t(A)$ converge to a proper limit as $t \to \infty$? Why or why not? If so, what is that limit?

(e) What is the rate (per unit of time) at which the taxi makes trips departing from location A heading toward location B? Start by defining what is meant by "rate" here.

(f) What is the long-run conditional probability that the taxi will come next to location B, given that the taxi is now traveling away from location A?

(g) What is the long-run proportion of time that the taxi is traveling from A to C and the remaining time before getting to C is at least 30 minutes?

(h) Which of the previous answers would change if the travel times were changed from uniform to exponential with the same mean? (You need not do any new computations?)

(i) Suppose that the travel times are indeed changed from uniform to exponential with the same mean. Let $X(t)$ be the state of the taxi at time t, e.g., idle at A or traveling from A to B. Give an explicit formula (not numerical value) for the conditional probability

$$P(X(2) = \text{idle at}\, B \quad \text{and} \quad X(7) = \text{idle at}\, C | X(0) = \text{idle at}\, A).$$

3. Birth-and-Death (BD) Processes (30 points)

(a) Prove or disprove: For every irreducible finite-state BD process, we can express the transition probability matrix function $P(t) = (P_{i,j}(t))$ as

$$P(t) = \tilde{P}_{N(t)}, \quad t \geq 0.$$

where $\tilde{P}$ is the transition matrix of a finite-state discrete-time Markov chain (DTMC) and $\{N(t) : t \geq 0\}$ is a Poisson process that is independent of the DTMC.

(b) Prove or disprove: For every irreducible finite-state BD process, we can express the transition probabilities as

$$P_{i,j}(t) = \sum_{n=0}^{\infty} \frac{a_{i,j}(n)t^n}{n!}, \quad t \geq 0,$$

where $a_{i,j}(n)$ are real numbers depending on n, the birth rates λ_k and the death rates μ_k (but are independent of t).

(c) Prove or disprove: For every irreducible finite-state BD process, we can express the transition probabilities as

$$P_{i,j}(t) = b_j + \sum_{k=1}^{m-1} a_k e^{-r_k t}, \quad t \geq 0,$$

where b_j, a_k and r_k are real numbers with $b_j > 0$ for all j and $r_k > 0$ for all k, with b_j depending on j, the birth rates and the death rates (but is independent of i and t), while a_k depends on i, j, the birth rates and the death rates (but is independent of t).

(d) Prove or disprove: For every irreducible infinite-state BD process and initial state i, the expected time until the process makes k transitions is finite for all k.

(e) Prove or disprove: For every irreducible infinite-state BD process and initial state i, the expected time until the process makes infinitely many transitions is infinite.