IEOR 6711: Stochastic Models I, Professor Whitt

Solutions to Homework Assignment 10

Numerical Problems 1.(a) $\pi_5 = 0.1667$

1.(b) Yes, because the Markov chain is irreducicle and has a finite state space. The stationary probability of being in state 5 is $\pi_5 = 0.1667$. The stationary probability vector is π such that $\pi = \pi P$. However, there is no limiting probability (i.e., we do not have a limit for P^n as $n \to \infty$), because the chain is periodic, with period 2.

1.(c) For large n, $P_{1,5}^{2n+1} = 0$ and $P_{1,5}^{2n} \simeq 2\pi_5 = 0.3334$

1.(d) $1/\pi_5 = 6$

2.(a) $M_1 = 14.26303$

2.(b) $N_{1,5} = 2.21054$

2.(c) $B_{1,10} = 0.3684$

Problem 4.18 Let $a_j = e^{-\lambda}\lambda^j/j!$, $\quad j \geq 0$.

(a)

$$P_{0,j} = a_j \ , \ j < N, \quad P_{0,N} = 1 - \sum_{j=0}^{N-1} a_j$$

$$\text{For } i > 0, \ P_{i,j} = a_{j-i+1} \ , \ j = i-1, \cdots, N-1, \ P_{i,N} = 1 - \sum_{j=0}^{N-i} a_j \ .$$

(b) Yes, because it is a finite, irreducible Markov chain.

(c) As one of the equations is redundant, we can write them as follows :

$$\begin{aligned} \pi_j &= \pi_0 a_j + \sum_{i=1}^{j+1} \pi_i a_{j-i+1} \ , \quad j = 0, \cdots, N-1 \\ \sum_{j=0}^{N} \pi_j &= 1 \ . \end{aligned}$$

Problem 4.19 (a) are from state i to state j.

(b) go from a state in A to one in A^c.

(c) This follows because between any two transitions that go from a state in A to one in A^c there must be a transition from a state in A^c to one in A, and vice-versa.

(d) It follows from (c) that the long-run proportion of transitions that are from a state in A to one in A^c must equal the long-run proportion of transitions that go from a state in A^c to one in A; and that is what (d) asserts.

Problem 4.31 Let the states be

0 : spider and fly at same location

1 : spider at location 1 and fly at 2

2 : spider at 2 and fly at 1

$$P = \begin{bmatrix} 1 & 0 & 0 \\ .54 & .28 & .18 \\ .54 & .18 & .28 \end{bmatrix}$$

(a)

$$P_{11}^n = (0.46)^n \left[\frac{1}{2} + \frac{1}{2}\left(\frac{28}{23} - 1\right)^n\right]$$

which is obtained by first conditioning on the event that 0 is not entered and then using the fact that for the

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

chain $P_{00}^n = \frac{1}{2} + \frac{1}{2}(2p-1)^n$.

More generally, we can find explicit analytical expressions for n-step transition probabilities by applying the spectral representation of the sub-probability transition matrix

$$Q = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

(The same argument applies without that special structure. See the Appendix of Karlin and Taylor for a textbook review of this part of basic linear algebra.) We want to find constants λ such that

$$xQ = \lambda x \ . \quad (1)$$

Those are the eigenvalues of Q. To find the eigenvalues, we solve the equation

$$det(Q - \lambda I) = 0 \ ,$$

where det is the determinant. Here the equation is

$$(a - \lambda)^2 - b^2 = 0 \ ,$$

which yields two solutions: $a + b$ and $a - b$. We then find the left eigenvectors of Q. A row vector x is a left eigenvector of Q associated with the eigenvalue

λ if equation (1) hold. Similarly, the transpose of x, denoted by x^T, is a right eigenvector of Q associated with eigenvalue λ if

$$Qx^T = \lambda x^T \ . \tag{2}$$

We then can find a *spectral representation* for Q:

$$Q = R\Lambda L \ , \tag{3}$$

with the following properties: (i) R and L are square matrices with the same dimension as Q, (ii) the columns of R are right eigenvectors of Q; (iii) the rows of L are left eigenvectors of Q, (iv) $RL = LR = I$, and (v) Λ is a square diagonal matrix with the eigenvalues for its diagonal elements. As a consequence, we have

$$Q^n = R\Lambda^n L \quad \text{for all} \quad n \geq 1 \ , \tag{4}$$

enabling us to compute Q^n, easily because Λ^n is a diagonal matrix with diagonal elements λ^n, where λ is an eigenvector.

Here we get eigenvalues of Q equal to $a + b$ and $a - b$. Here we get eigenvector matrices

$$L = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

We obtain one of these by directly solving for the eigenvectors (which are not unique). Given L or R, we can obtain the other by inverting the matrix, i.e., $L = R^{-1}$.
Hence, equation (4) holds

$$Q^n = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \begin{bmatrix} (a+b)^n & 0 \\ 0 & (a-b)^n \end{bmatrix} \times \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Thus, in general,

$$Q^n_{1,1} = \frac{(a+b)^n}{2} + \frac{(a-b)^n}{2}$$

and, in particular,

$$Q^n_{1,1} = \frac{(0.46)^n}{2} + \frac{(0.10)^n}{2}$$

(b) $\mathsf{E}[N] = \frac{1}{.54}$ since N is geometric (on the positive integers, not including 0) with $p = 0.54$.