IEOR 6711: Stochastic Models I

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Solutions to Homework Assignment 2

Problem 1.6 Answer in back.

Problem 1.7 Imagine that there are k buckets and we choose k balls and fill each bucket at the same time. Set

$$I_i = \begin{cases} 1 & \text{if } i\text{-th bucket contains a white ball} \\ 0 & \text{if } i\text{-th bucket contains a black ball.} \end{cases}$$

Then $X = \sum_{i=1}^{k} I_i$. Observe that for $i \neq j$,

$$\begin{aligned} \mathbf{E}[I_i] &= \mathbf{E}[I_i^2] = \mathbf{P}(I_i = 1) = \frac{n}{n+m}, \\ \mathbf{E}[I_i|I_j = 1] &= \mathbf{P}(I_i = 1|I_j = 1) = \frac{n-1}{n+m-1} \text{ and} \\ \mathbf{E}[I_iI_j] &= \mathbf{E}[\mathbf{E}[I_iI_j|I_j]] = \mathbf{E}[I_j\mathbf{E}[I_i|I_j]] = \mathbf{P}(I_j = 1)\mathbf{E}[I_i|I_j = 1] = \frac{n}{n+m} \times \frac{n-1}{n+m-1} . \end{aligned}$$

Using these, we get

$$\begin{aligned} \mathbf{E}[X] &= k\mathbf{P}(I_1 = 1) = \frac{kn}{n+m} , \\ \mathbf{E}[X^2] &= k\mathbf{E}[I_1^2] + k(k-1)\mathbf{E}[I_1I_2] \\ &= \frac{kn}{n+m} + \frac{k(k-1)n(n-1)}{(n+m)(n+m-1)} , \text{ and} \\ \mathbf{V}(X) &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \frac{knm(n+m-k)}{(n+m)^2(n+m-1)}. \end{aligned}$$

Problem 1.9 Assume that the results of each pairing are independent with each of the players being equally likely to win. For each permutation $i_1, \cdots, i_n$ of $1, 2, \cdots, n$ define an indicator variable $I_{(i_1,\cdots,i_n)}$ equal to 1 if that permutation is a Hamiltonian and 0 if it is not. Then

$$\begin{aligned} \mathbf{E}[\text{Number of hamiltonians}] &= \mathbf{E}[\textstyle\sum I_{(i_1,\cdots,i_n)}] \\ &= n!\mathbf{E}[I_{(1,2,\cdots,n)}] \\ &= \frac{n!}{2^{n-1}} . \end{aligned}$$

Hence, for at least one outcome the number of Hamiltonians must be at least $\frac{n!}{2^{n-1}}$.

Problem 1.14 Answer in back.

(a) Some explanation might help. The solutions in the back are somewhat confusing. There is an extra "is 1" in the third line. It first is good to observe that

$$X_1 = \sum_{i=1}^{10} Y_i \ ,$$

where Y_i are IID. (Even Y_1 is well defined this way.) Then a step is left out. Let p be the probability a 1 occurs before an even number. We develop an equation for p:

$$p = \frac{1}{6} + \frac{2}{6}p \ ,$$

so that

$$p = \frac{1}{4}.$$

Then use p to calculate $E[Y_i]$ by developing another equation. That step is given in the answers.

Problem 1.17 Answer in back.

Problem 1.18 Let N be the number of flips that are made until a string of r heads in a row. Define T as the the number of trials until the first tails. Then we have

$$\mathbf{E}[N|T=k] = \begin{cases} k + \mathbf{E}[N] & \text{if } k \le r \\ r & \text{if } k > r \ . \end{cases}$$

Using the fact that T has geometric distribution,

$$\begin{aligned}
\mathbf{E}[N] &= \mathbf{E}[\mathbf{E}[N|T]] = \sum_{k=1}^{\infty} \mathbf{E}[N|T=k]\mathbf{P}(T=k) \\
&= \sum_{k=1}^{r}(k+\mathbf{E}[N])(1-p)p^{k-1} + \sum_{k=r+1}^{\infty} r(1-p)p^{k-1} \\
&= (1-p)\sum_{k=1}^{r}(k+\mathbf{E}[N])p^{k-1} + r(1-p)\sum_{k=r+1}^{\infty} p^{k-1} \\
&= (1-p)\sum_{k=1}^{r} kp^{k-1} + \mathbf{E}[N](1-p)\sum_{k=1}^{r} p^{k-1} + rp^r \\
&= \frac{1-(r+1)p^r + rp^{r+1}}{1-p} + \mathbf{E}[N](1-p^r) + rp^r \\
&= \frac{1-p^r}{1-p} + \mathbf{E}[N](1-p^r) \\
&= \frac{1-p^r}{(1-p)p^r} \ .
\end{aligned}$$

Problem 1.20 Let L be the left hand point of the first interval. Note that $\{N(x)|L = y\} = \{1 + N(y) + N(x - y - 1)\}$. If $x > 1$,

$$\begin{aligned}
M(x) &= \mathbf{E}[N(x)] = \mathbf{E}[\mathbf{E}[N(x)|L]] \\
&= \int_0^{x-1} \mathbf{E}[N(x)|L = y] \frac{dy}{x-1} \\
&= \frac{1}{x-1} \int_0^{x-1} \mathbf{E}[1 + N(y) + N(x - y - 1)]dy \\
&= 1 + \frac{1}{x-1} \int_0^{x-1} \mathbf{E}[N(y) + N(x - y - 1)]dy \\
&= 1 + \frac{1}{x-1} \int_0^{x-1} (M(y) + M(x - y - 1))dy \\
&= 1 + \frac{2}{x-1} \int_0^{x-1} M(y)dy.
\end{aligned}$$

Problem 1.28 The MGF is given by $\phi(t) = \frac{\lambda}{\lambda - t}$. So

$$\phi'(t) = \frac{\lambda}{(\lambda - t)^2} \quad \text{and} \quad \phi''(t) = \frac{2\lambda}{(\lambda - t)^3} \ .$$

Hence,

$$\begin{aligned}
\mathbf{E}[X] &= \phi'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} \\
\mathbf{V}(X) &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 \\
&= \phi''(0) - \frac{1}{\lambda^2} = \frac{2\lambda}{\lambda^3} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2} \ .
\end{aligned}$$

Problem 1.29 The MGF of an exponential random variable, X, is $\phi_X(t) = \frac{\lambda}{\lambda - t}$. Then

$$\begin{aligned}
\phi_{\sum_n X_i}(t) &= \mathbf{E}[e^{t \sum_n X_i}] \\
&= \prod_{i=1}^{n} \mathbf{E}[e^{tX_i}] = \phi_X(t)^n \\
&= \left(\frac{\lambda}{\lambda - t} \right)^n
\end{aligned}$$

which is an MGF of an Gamma distribution with parameter (n, λ). Hence the result follows from the uniqueness of MGF.

Problem 1.31

$$\mathbf{P}(\min\{X, Y\} > a | \min\{X, Y\} = X) = \mathbf{P}(X > a | X < Y) = \frac{\mathbf{P}(a < X, X < Y)}{\mathbf{P}(X < Y)} \ .$$

$$\begin{aligned}
\mathbf{P}(a < X, X < Y) &= \int_a^\infty \mathbf{P}(Y > X | X = x)\lambda_1 e^{-\lambda_1 x} dx = \int_a^\infty e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x} dx \\
&= \lambda_1 \int_a^\infty e^{-(\lambda_1+\lambda_2)x} dx = \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2} a \ ,
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}(X < Y) &= \int_0^\infty \mathbf{P}(Y > X | Y = y)\lambda_2 e^{-\lambda_2 y} dy = \int_0^\infty (1 - e^{-\lambda_1 y})\lambda_2 e^{-\lambda_2 y} dy \\
&= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2} \ .
\end{aligned}$$

Hence,

$$\mathbf{P}(\min\{X, Y\} > a | \min\{X, Y\} = X) = e^{-(\lambda_1+\lambda_2)a} \ .$$

Problem 1.34 Answer in back.

Problem 1.43

$$\mathbf{P}(X \geq a) = \mathbf{P}(X^t \geq a^t) \leq \frac{\mathbf{E}[X^t]}{a^t}$$

with the inequality following from the Markov inequality. Let X be exponential with rate 1, and let $a = t = n$ in the preceding, to obtain that

$$e^{-n} \leq \frac{n!}{n^n} \ .$$

(Of course, the above inequality could also be shown by noting that it is equivalent to the statement that $\mathbf{P}(Y = n) \leq 1$ where Y is Poisson with mean n.

Problem 1.23 Let $i \to j$ be the event that the particle moves from i to j in one step. Let $i \Rightarrow j$ be the event that the particle ever reaches j starting i. Conditioning on the random variable denoting the first movements of the particle, I,

(a)

$$\begin{aligned}
\alpha &= \mathbf{P}(0 \Rightarrow 1) \\
&= \mathbf{E}[\mathbf{P}(0 \Rightarrow 1 | I)] \\
&= \mathbf{P}(0 \to 1)\mathbf{P}(1 \Rightarrow 1) + \mathbf{P}(0 \to -1)\mathbf{P}(-1 \Rightarrow 1) \\
&= p \times 1 + (1-p)\mathbf{P}(-1 \Rightarrow 1) \\
&= p + (1-p)\mathbf{P}(-1 \Rightarrow 0, 0 \Rightarrow 1) \\
&= p + (1-p)\mathbf{P}(-1 \Rightarrow 0)\mathbf{P}(0 \Rightarrow 1) \\
&= p + (1-p)\mathbf{P}(0 \Rightarrow 1)^2 \\
&= p + (1-p)\alpha^2 \ .
\end{aligned}$$

(b) Solving the previous quadratic equation, we get two solutions, 1 and $\frac{p}{1-p}$. The condition $\frac{p}{1-p} < 1$ implies $p < 1/2$. Hence if $p \geq 1/2$, α should be 1. For $p < 1/2$, the *strong law of large numbers* says that the particle ever goes to the negative infinity with probability 1. If $\alpha = 1$, then the starting position would be reached infinitely often, which contradicts to the *strong law of large numbers*. Hence

$$\alpha = \begin{cases} 1 & \text{if } p \geq 1/2 \\ \frac{p}{1-p} & \text{if } p < 1/2 \ . \end{cases}$$

(c)

$$\begin{aligned} \mathbf{P}(0 \Rightarrow n) &= \mathbf{P}(0 \Rightarrow 1) \times \cdots \times \mathbf{P}(n-1 \Rightarrow n) \\ &= \mathbf{P}(0 \Rightarrow 1) \times \cdots \times \mathbf{P}(0 \Rightarrow 1) \\ &= \mathbf{P}(0 \Rightarrow 1)^n \\ &= \alpha^n \ . \end{aligned}$$

(d)

$$\begin{aligned} \mathbf{P}(i \to i+1 | i \Rightarrow n) &= \frac{\mathbf{P}(i \to i+1, i \Rightarrow n)}{\mathbf{P}(i \Rightarrow n)} \\ &= \frac{\mathbf{P}(i \Rightarrow n | i \to i+1)\mathbf{P}(i \to i+1)}{\mathbf{P}(i \Rightarrow n)} \\ &= \frac{\mathbf{P}(i+1 \Rightarrow n)p}{\mathbf{P}(i \Rightarrow n)} \\ &= \frac{\alpha^{n-i-1}p}{\alpha^{n-i}} \\ &= \frac{p}{\alpha} \\ &= 1-p \ . \end{aligned}$$

Problem 1.24 Let $T_{(i \Rightarrow j)}$ the number of steps to reach j first time starting i. Then we have an apparent arithmetic like $T_{(-1 \Rightarrow 1)} = T_{(-1 \Rightarrow 0)} + T_{(0 \Rightarrow 1)}$ and a distributional identity like $T_{(-1 \Rightarrow 0)} \stackrel{d}{=} T_{(0 \Rightarrow 1)}$. We also know that $T_{(-1 \Rightarrow 0)}$ and $T_{(0 \Rightarrow 1)}$ are independent because of the independence of every transition. That is, $T_{(-1 \Rightarrow 0)}$ and $T_{(0 \Rightarrow 1)}$ are iid. Using the notation in the book, $T \equiv T_{(0 \Rightarrow 1)}$,

$$\begin{aligned} \mathbf{E}[T_{(-1 \Rightarrow 1)}] &= 2\mathbf{E}[T] \ , \\ \mathbf{V}(T_{(-1 \Rightarrow 1)}) &= 2\mathbf{V}(T) \ . \end{aligned}$$

Let the random variable X denote the particle's location after the first move.

(a) Conditioning on X gives

$$\begin{aligned}
\mathbf{E}[T] &= \mathbf{E}[\mathbf{E}[T|X]] \\
&= \mathbf{E}[T|X=1]\mathbf{P}(X=1) + \mathbf{E}[T|X=-1]\mathbf{P}(X=-1) \\
&= 1 \times p + (1 + \mathbf{E}[T_{(-1\Rightarrow 1)}])(1-p) \\
&= 1 + 2(1-p)\mathbf{E}[T] \ .
\end{aligned}$$

Hence, $\mathbf{E}[T] = \infty$ if $p \leq 1/2$. If we can show that $\mathbf{E}[T] < \infty$ when $p > 1/2$, we obtain in this case that

$$\mathbf{E}[T] = \frac{1}{2p-1} \ .$$

Now let's show that $\mathbf{E}[T] < \infty$ if $p > 1/2$: Let $p^{(n)}$ denotes the probability that the particle reaches 1 by n-transitions starting 0. Then n should be odd. That is, only $p^{(2n+1)}$ is nonzero. Now we have an upper bound on this probability:

$$p^{(2n+1)} \leq \binom{2n}{n} p[p(1-p)]^n \sim p\frac{[4p(1-p)]^n}{\sqrt{\pi n}}$$

using an approximation, due to Stirling, which asserts that

$$n! \sim n^{n+1/2}e^{-n}\sqrt{2\pi}$$

where $a_n \sim b_n$ denotes $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Now it is easy to verify that if $a_n \sim b_n$, then $\sum_n a_n < \infty$ if, and only if, $\sum_n b_n < \infty$. Hence $\mathbf{E}[T] < \infty$ if

$$\sum_{n=0}^{\infty} (2n+1)p\frac{[4p(1-p)]^n}{\sqrt{\pi n}} < \infty$$

which is true if $4p(1-p) < 1$ or $p \neq 1/2$.

(b) Noting that $T|\{X=1\} = 1$ and $T|\{X=-1\} = 1 + T_{(-1\Rightarrow 1)}$ which is 1 plus the *convolution* of two independent random variables both having the distribution of T. Therefore,

$$\mathbf{E}[T|X=1] = 1, \quad \mathbf{E}[T|X=-1] = 1 + 2\mathbf{E}[T]$$
$$\mathbf{V}(T|X=1) = 0, \quad \mathbf{V}(T|X=-1) = 2\mathbf{V}(T)$$

and thus

$$\begin{aligned}
\mathbf{V}(\mathbf{E}[T|X]) &= \mathbf{V}(\mathbf{E}[T|X]-1) = 4\mathbf{E}[T]^2 p(1-p) = \frac{4p(1-p)}{(2p-1)^2} \\
\mathbf{E}[\mathbf{V}(T|X)] &= 2(1-p)\mathbf{V}(T) \ .
\end{aligned}$$

By the conditional variance formula

$$\mathbf{V}(T) = 2(1-p)\mathbf{V}(T) + \frac{4p(1-p)}{(2p-1)^2}$$

which gives the result.

(c) $T_{(0\Rightarrow n)} = T_{(0\Rightarrow 1)} + \cdots + T_{(n-1\Rightarrow n)} = \sum_{i=1}^{n} T_i$ where T_i are iid having distribution of T. Hence

$$\mathbf{E}[T_{(0\Rightarrow n)}] = n\mathbf{E}[T] \ .$$

(d) By the same reasoning as in (c),

$$\mathbf{V}(T_{(0\Rightarrow n)}) = n\mathbf{V}(T) \ .$$