# IEOR 6711: Stochastic Models I

## Professor Whitt

## Solutions to Homework Assignment 3

**Problem 2.1** The conditions (i) and (ii) of definition 2.1.2 are apparent from the definition 2.1.1. Hence it is sufficient to show that definition 2.1.1 implies the last two conditions of definition 2.1.2.

- $\mathbf{P}(N(h) = 1) = \lambda h + o(h)$ :

$$\lim_{h \to 0} \frac{\mathbf{P}(N(h) = 1) - \lambda h}{h} = \lim_{h \to 0} \frac{e^{-\lambda h}\lambda h - \lambda h}{h} = \lim_{h \to 0} \left(e^{-\lambda h} - 1\right)\lambda = 0 \ .$$

- $\mathbf{P}(N(h) \geq 2) = o(h)$ :

$$\lim_{h \to 0} \frac{\mathbf{P}(N(h) \geq 2)}{h} = \lim_{h \to 0} \frac{1 - e^{-\lambda h} - e^{-\lambda h}\lambda h}{h} = \lim_{h \to 0} \frac{1 - e^{-\lambda h}}{h} - \lim_{h \to 0} e^{\lambda h}\lambda$$
$$= \frac{1 - (1 - \lambda h + o(h))}{h} - (1 - o(h))\lambda = \lambda + \frac{o(h)}{h} - \lambda + o(h) \to 0.$$

Or using $e^{ax} = 1 + ax + o(x)$, $o(x) \times o(x) = o(x)$, and $f(x) \times o(x) = o(x)$ for any $f(x)$ satisfying $\lim_{x \to 0} f(x)$ is finite,

- $\mathbf{P}(N(h) = 1) = e^{-\lambda h}\lambda h = (1 - \lambda h + o(h))\lambda h = \lambda h + o(h)$ .
- $\mathbf{P}(N(h) \geq 2) = 1 - e^{-\lambda h} - e^{-\lambda h}\lambda h = 1 - (1 + \lambda h)(1 - \lambda h + o(h)) = \lambda^2 h^2 + o(h) = o(h)$ .

**Problem 2.2** For $s < t$,

$$\begin{aligned}
\mathbf{P}(N(s) = k | N(t) = n) &= \frac{\mathbf{P}(N(s) = k, N(t) = n)}{\mathbf{P}(N(t) = n)} = \frac{\mathbf{P}(N(s) = k, N(t) - N(s) = n - k)}{\mathbf{P}(N(t) = n)} \\
&= \frac{\mathbf{P}(N(s) = k)\mathbf{P}(N(t) - N(s) = n - k)}{\mathbf{P}(N(t) = n)} \\
&= \frac{\mathbf{P}(N(s) = k)\mathbf{P}(N(t - s) = n - k)}{\mathbf{P}(N(t) = n)} \\
&= \left(\frac{e^{-\lambda s}(\lambda s)^k}{k!}\right)\left(\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{(n-k)}}{(n-k)!}\right)\left(\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right)^{-1} \\
&= \frac{n!}{k!(n-k)!}\frac{s^k(t-s)^{n-k}}{t^n} \\
&= \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k} \ .
\end{aligned}$$

**Problem 2.4** Let $\{X(t) : t \geq 0\}$ be a stochastic process having stationary independent increments and $X(0) = 0$. (People call it *Levy* process.) Two typical *Levy* processes are *Poisson* and *Brownian motion* processes. They are representatives of purely discrete and purely continuous continuous time stochastic processes, respectively. Furthermore, it is not easy to find any non-trivial Levy process except them. Now let's try to express $\mathbf{E}[X(t)X(t+s)]$ by moments of $X$ using only the properties of Levy process.

$$\begin{aligned}
\mathbf{E}[X(t)X(t+s)] &= \mathbf{E}[X(t)(X(t+s) - X(t) + X(t))] \\
&= \mathbf{E}[X(t)(X(t+s) - X(t)) + X(t)^2] \\
&= \mathbf{E}[X(t)(X(t+s) - X(t))] + \mathbf{E}[X(t)^2] \\
&= \mathbf{E}[X(t)]\mathbf{E}[(X(t+s) - X(t))] + \mathbf{E}[X(t)^2] \quad \text{by independent increment} \\
&= \mathbf{E}[X(t)]\mathbf{E}[(X(s))] + \mathbf{E}[X(t)^2] \quad \text{by stationary increment}
\end{aligned}$$

Now return to our original process, Poisson process. By substituting $\mathbf{E}[N(t)] = \lambda t$, $\mathbf{E}[N(t)^2] = \lambda t + (\lambda t)^2$,

$$\mathbf{E}[N(t)N(t+s)] = \lambda^2 st + \lambda t + \lambda^2 t^2 \ .$$

A digression : if $X(t) \sim \text{Normal}(0, t)$, what is the result? This is the Brownian motion case.

**Problem 2.5**

- $\{N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.
  Axioms (i) and (ii) of definition e.1.2 easily follow. Letting $N(t) = N_1(t) + N_2(t)$,

$$\begin{aligned}
\mathbf{P}(N(h) = 1) &= \mathbf{P}(N_1(h) = 1, N_2(h) = 0) + \mathbf{P}(N_1(h) = 0, N_2(h) = 1) \\
&= \lambda_1 h(1 - \lambda_2 h) + \lambda_2 h(1 - \lambda_1 h) + o(h) \\
&= (\lambda_1 + \lambda_2)h + o(h)
\end{aligned}$$

  and

$$\begin{aligned}
\mathbf{P}(N(h) = 2) &= \mathbf{P}(N_1(h) = 1, N_2(h) = 1) \\
&= (\lambda_1 h + o(h))(\lambda_2 h + o(h)) \\
&= \lambda_1 \lambda_2 h^2 + o(h) = o(h) \ .
\end{aligned}$$

- The probability that the first event of the combined process comes from $\{N_1(t), t \geq 0\}$ is $\lambda_1/(\lambda_1 + \lambda_2)$, independently of the time of the event. Let $X_i$ and $Y_i$ are the $i$-th inter arrival times of $N_1$ and $N_2$, respectively. Then

$$\begin{aligned}
\mathbf{P}(\text{first from } N_1 | \text{first at } t) &= \mathbf{P}(X_1 < Y_1 | \min\{X_1, Y_1\} = t) \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

  where the last equality comes from our old homework 1.1.34.

**Problems 2.6–2.9** Answers in back of the book.

**Problem 2.10** **(a)** First note that the time until next bus arrival follows exponential distribution with rate $\lambda$ by the *memoryless* property of exponential distribution. Let $X$ be the time until next bus arrival. Then $T$, the random variable representing the time spent to reach home is

$$\begin{aligned} T &= \begin{cases} X+R & \text{if } X \le s \ , \\ s+W & \text{if } X > s \end{cases} \\ &= (X+R)\mathbf{1}_{\{X\le s\}} + (s+W)\mathbf{1}_{\{X>s\}} \ . \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{E}[T] &= \mathbf{E}[(X+R)\mathbf{1}_{\{X\le s\}}] + \mathbf{E}[(s+W)\mathbf{1}_{\{X>s\}}] \\ &= \mathbf{E}[X\mathbf{1}_{\{X\le s\}}] + R\mathbf{E}[\mathbf{1}_{\{X\le s\}}] + (s+W)\mathbf{E}[\mathbf{1}_{\{X>s\}}] \\ &= \int_0^s x\lambda e^{-\lambda x}dx + R\mathbf{P}(X\le s) + (s+W)\mathbf{P}(X>s) \\ &= \int_0^s x\lambda e^{-\lambda x}dx + R(1-e^{-\lambda s}) + (s+W)e^{-\lambda s} \\ &= -xe^{-\lambda x}\Big|_0^s - \frac{1}{\lambda}e^{-\lambda x}\Big|_0^s + R + (s+W-R)e^{-\lambda s} \\ &= \frac{1}{\lambda}(1-e^{-\lambda s}) - se^{-\lambda s} + R + (s+W-R)e^{-\lambda s} \\ &= \frac{1}{\lambda} + R + \left(W - R - \frac{1}{\lambda}\right)e^{-\lambda s}. \end{aligned}$$

**(b)** Considering

$$\frac{d}{ds}\mathbf{E}[T] = (1-\lambda(W-R))e^{-\lambda s} \begin{cases} >0 & \text{if } W < \frac{1}{\lambda}+R \ , \\ =0 & \text{if } W = \frac{1}{\lambda}+R \ , \\ <0 & \text{if } W > \frac{1}{\lambda}+R \ , \end{cases}$$

we get

$$\text{argmin}_{0\le s<\infty}\mathbf{E}[T] = \begin{cases} 0 & \text{if } W < \frac{1}{\lambda}+R \ , \\ \text{any number } \in [0,\infty) & \text{if } W = \frac{1}{\lambda}+R \ , \\ \infty & \text{if } W > \frac{1}{\lambda}+R \ , \end{cases}$$

**(c)** Since the time until the bus arrives is exponential, it follows by the *memoryless* property that if it is optimal to wait any time then one should always continue to wait for the bus.

**Problem 2.11** Conditioning on the time of the next car yields

$$\mathbf{E}[\text{wait}] = \int_0^\infty \mathbf{E}[\text{wait}|\text{car at } x]\lambda e^{-\lambda x}dx \ .$$

Now,

$$\mathbf{E}[\text{wait}|\text{car at } x] = \begin{cases} x + \mathbf{E}[\text{wait}] & \text{if } x < T \ , \\ 0 & \text{if } x \geq T \end{cases}$$

and so

$$\mathbf{E}[\text{wait}] = \int_0^T x\lambda e^{-\lambda x} dx + \mathbf{E}[\text{wait}](1 - e^{-\lambda T})$$

or

$$\mathbf{E}[\text{wait}] = \frac{1}{\lambda} e^{-\lambda T} - T - \frac{1}{\lambda} \ .$$

**Problem 2.13** First note that $T$ is (unconditionally) exponential with rate $\lambda p$, $N$ is geometric with parameter $p$, and the distribution of $T$ given that $N = n$ is gamma with parameter $n$ and $\lambda$, we obtain

$$\begin{aligned} \mathbf{P}(N = n | t - \epsilon < T \leq t) &= \frac{\mathbf{P}(t - \epsilon < T \leq t | N = n)}{\mathbf{P}(t - \epsilon < T \leq t)} \\ &\simeq \frac{\epsilon \lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \frac{p(1-p)^{n-1}}{\epsilon \ \lambda p \ e^{-\lambda p t}} \\ &= \frac{e^{-\lambda t(1-p)} [\lambda t (1-p)]^{n-1}}{(n-1)!} \ . \end{aligned}$$

Hence, given that $T = t$, $N$ has the distribution of $X + 1$, where $X$ is a Poisson random variable with mean $\lambda t(1-p)$. A simpler argument is to note that the occurrences of failure causing shocks and non-failure causing shocks are independent Poisson processes. Hence, the number of non-failure causing shocks by time $t$ is Poisson with mean $\lambda(1-p)t$, independent of the event that the first failure shock occurred at that time.