IEOR 6711: Stochastic Models I, Professor Whitt

Solutions to Homework Assignment 4.

Problem 2.14 Let us use D_j instead of O_j for the number of people getting off at floor j. Let $D_{i,j}$ denote the number of people that get on at floor i and get off at floor j. First, $D_{i,j}$ is an independent thinning of N_i with

$$N_i = \sum_{j=i+1}^{n} D_{i,j}$$

so $D_{i,j}$ for different j are independent Poisson random variables. But then N_i are independent for different i. Consequently $D_{i,j}$, as i and j both vary, are independent Poisson random variables, with mean $\lambda_i p_{i,j}$.

(a)-(c) Clearly,

$$D_j = \sum_{i=0}^{j-1} D_{i,j}$$

so that D_j is the sum of independent Poisson random variables, so itself must have a Poisson distribution with mean

$$E[D_j] = \sum_{i=0}^{j-1} E[D_{i,j}] = \sum_{i=0}^{j-1} \lambda_i p_{i,j} \ .$$

Moreover, Since D_{j_1} and D_{j_2} for $j_1 \neq j_2$ have no variables in common in the sums, these two Poisson random variables are independent Poisson random variables.

Problem 2.16 For fixed j, let

$$I_i = \begin{cases} 1 & \text{if outcome } i \text{ occurs } j \text{ times,} \\ 0 & \text{otherwise ,} \end{cases}$$

and note that $I_i, i = 1, \cdots, n$ are independent since the number of type i outcomes, $i = 1, \cdots, n$, will be independent. (If we think that there is a Poisson process with rate λ and we count on $[0, 1]$, then the n-types of events are independent by proposition 2.3.2 and hence so are I_i's.) Writing

$$X_j = \sum_{i=1}^{n} I_j$$

we have

$$\mathbf{E}[X_j] = \sum_{i=1}^{n} \mathbf{P}(I_i = 1)$$

and

$$\mathbf{V}ar[X_j] = \sum_{i=1}^{n} \mathbf{P}(I_i = 1)(1 - \mathbf{P}(I_i = 1)) \ .$$

As the number of times outcome i results is Poisson with mean λP_i we have that

$$\mathbf{P}(I_i = 1) = \frac{e^{-\lambda P_i}(\lambda P_i)^j}{j!}$$

and so

$$\mathbf{E}[X_j] = \sum_{i=1}^{n} \frac{e^{-\lambda P_i}(\lambda P_i)^j}{j!} ,$$

$$\mathbf{V}ar[X_j] = \mathbf{E}[X_j] - \sum_{i=1}^{n} \frac{e^{-2\lambda P_i}(\lambda P_i)^{2j}}{(j!)^2} .$$

Problem 2.17 **(a)** $\{X_{(i)} = x\}$ implies $i-1$ X_j's are less than x and $n-i$ X_j's are greater than x and one is equal to x. Hence

$$\begin{aligned} f_{X_{(i)}}(x) &= \frac{\mathbf{P}(X_{(i)} \in (x, x+dx))}{dx} \\ &= \frac{n!}{(i-1)!1!(n-i)!} \frac{(F(x))^{i-1} f(x) dx (\bar{F}(x+dx))^{n-i}}{dx} \\ &= \frac{n!}{(i-1)!(n-i)!}(F(x))^{i-1} f(x) (\bar{F}(x))^{n-i} . \end{aligned}$$

(b) At least i.

(c)

$$\begin{aligned} \mathbf{P}(X_{(i)} \le x) &= \mathbf{P}(i \text{ or more } X_j's \text{ are less than or equal to } x) \\ &= \sum_{k=i}^{n} \binom{n}{k} (F(x))^k (\bar{F}(x))^{n-k} . \end{aligned}$$

(d)

$$\begin{aligned} \mathbf{P}(X_{(i)} \le x) &= \sum_{k=i}^{n} \binom{n}{k} (F(x))^k (\bar{F}(x))^{n-k} \\ &= \int_0^x f_{X_{(i)}}(t) dt = \int_0^x \frac{n!}{(i-1)!(n-i)!}(F(t))^{i-1} f(t) (\bar{F}(t))^{n-i} dt \\ &= \int_0^x \frac{n!}{(i-1)!(n-i)!}(F(t))^{i-1} (\bar{F}(t))^{n-i} dF(t) \end{aligned}$$

and substituting $F(x)$ by y gives

$$\sum_{k=i}^{n} \binom{n}{k} y^k (1-y)^{n-k} = \int_0^y \frac{n!}{(i-1)!(n-i)!} x^{i-1} (1-x)^{n-i} dx .$$

(e) First note that $S_{(i)}|\{N(t)=n\} \sim \text{Uniform}(0,t)$ if $i \le n$. Hence from (a),

$$\begin{aligned}
\mathbf{E}[X_{(i)}] &= \int_0^t x f_{X_{(i)}}(x)dx \\
&= \int_0^t x \frac{n!}{(i-1)!(n-i)!}\left(\frac{x}{t}\right)^{i-1}\left(1-\frac{x}{t}\right)^{n-i}\frac{dx}{t} \\
&= \frac{i}{n+1}t\int_0^t \frac{(n+1)!}{i!(n-i)!}\left(\frac{x}{t}\right)^{i}\left(1-\frac{x}{t}\right)^{n-i}\frac{dx}{t} \\
&= \frac{i}{n+1}t
\end{aligned}$$

if $i \le n$. For $i > n$, using memoryless property

$$\begin{aligned}
\mathbf{E}[S_i|N(t)=n] &= t+\mathbf{E}[S_i-t|N(t)=n] = t+\mathbf{E}[X_{n+1}+\cdots+X_i] \\
&= t+\frac{i-n}{\lambda}\ .
\end{aligned}$$

Hence

$$\mathbf{E}[S_i|N(t)=n] = \begin{cases} \frac{i}{n+1}t & \text{if } i \le n, \\ t+\frac{i-n}{\lambda} & \text{if } i > n\ . \end{cases}$$

Problem 2.18 As the joint density of $U_{(1)}, U_{(2)}, \cdots, U_{(n)}$ is $f(u_1,\cdots,u_n) = 1/n!$, $0 < u_1 < \cdots < u_n < 1$, the conditional density is

$$\begin{aligned}
f(u_1,\cdots,u_{n-1},y|U_{(n)}=y) &= \frac{f(u_1,\cdots,u_{n-1},y|U_{(n)}=y)}{f_{U_{(n)}(y)}} \\
&= \frac{n!}{ny^{n-1}} \\
&= \frac{(n-1)!}{y^{n-1}}\ , \quad 0 < u_1 < \cdots < u_{n-1} < y
\end{aligned}$$

which proves the result.

Problem 2.19 Observe that this is essentially an $M/G/\infty$ queue problem, so we can apply the "Physics" paper. We could also easily generalize the results to a nonhomogeneous Poisson arrival process.

It is not clear if the service times apply to each bus or to each customer. We assume the service times apply to the buses, rather than the customers, but the overall mean is unaffected by that result. Presumably the service times are meant to be IID. We need to assume that too.

(a) We first want to apply the splitting or thinning property. We can split the original Poisson process according to the number of customers on the bus. A bus is of type j if the bus contains j customers. Thus, by the splitting property, the overall arrival process is the superposition of infinitely many independent Poisson processes. Poisson process j has arrival rate $\lambda\alpha_j$. Thus the whole system behaves as infinitely many independent $M/G/\infty$ queues. Arrival process j has arrival rate $\lambda\alpha_j$.

By the basic $M/G/\infty$ theory, the number of buses of type j to depart from the system has a Poisson distribution with mean $m_j(t) = \lambda\alpha_j \int_0^t G(s)\, ds, \quad t \geq 0$. Thus the total number of *buses* to depart in $[0,t]$ also has a Poisson distribution with a mean equal to the sum of the means, i.e., $m(t) = \sum_{j=1}^{\infty} m_j(t)$.

However, we are asked about the number of *customers*. The number of customers on a bus of type j (which contains exactly j customers) is j times the Poisson random variable with mean $m_j(t)$. Thus the overall mean is

$$E[X(t)] = \sum_{j=1}^{\infty} j m_j(t) = \lambda \int_0^t G(s)\, ds \sum_{j=1}^{\infty} j\alpha_j \ . \tag{1}$$

(b) If the service times apply to buses, then batches of customers depart together, so that the departure process of customers cannot be Poisson. Even if the service times are associated with customers, the departure process is not Poisson. That is easy to show if the service time distribution G is in fact deterministic. Then the customers that arrive together on the same bus will also depart together. Otherwise the non-Poisson character of the departure process is harder to show. But since the arrival process is a batch Poisson process, we should not expect the departure process to be a Poisson process.

We now give an alternative direct derivation of part (a): Let N_i be the number of customers in $i-$th busload. Then $\mathbf{E}[N_i] = \sum_{k=1}^{\infty} k\alpha_k$ since $\mathbf{P}(N_i = k) = \alpha_k$. Let the indicator, $I_{i,j}$ for $1 \leq j \leq N_i$, denote whether the $j-th$ customer in $i-$th busload finishes his/her service at time t. Then $X(t) = \sum_{i=0}^{N(t)} \sum_{j=1}^{N_i} I_{i,j}$.

(a) First note that $\mathbf{E}[\mathbf{E}[X|Y,Z]|Z] = \mathbf{E}[X|Z]$ and $\mathbf{E}[\mathbf{E}[X|Z]|Y,Z] = \mathbf{E}[X|Z]$ which mean *smaller information wins always in double conditioning!* (You might prove it right now, or may consult the equation (6.1.2) in page 296.) Also, we have

$$\mathbf{E}[I_{i,j}|N(t) = n] = \int_0^t G(t-s)\frac{1}{t}ds = \frac{1}{t}\int_0^t G(s)ds \equiv p \ .$$

$$\begin{aligned}
\mathbf{E}[X(t)] &= \mathbf{E}\left[\sum_{i=0}^{N(t)} \sum_{j=1}^{N_i} I_{i,j}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\sum_{i=0}^{N(t)} \sum_{j=1}^{N_i} I_{i,j} \,\middle|\, N(t)\right]\right] \\
&= \mathbf{E}\left[\sum_{i=0}^{N(t)} \mathbf{E}\left[\sum_{j=1}^{N_i} I_{i,j} \,\middle|\, N(t)\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\left[\sum_{i=0}^{N(t)} \mathbf{E}\left[\mathbf{E}\left[\sum_{j=1}^{N_i} I_{i,j}\middle| N(t), N_i\right]\middle| N(t)\right]\right] \quad \text{(smaller information wins)} \\
&= \mathbf{E}\left[\sum_{i=0}^{N(t)} \mathbf{E}\left[\sum_{j=1}^{N_i} \mathbf{E}\left[I_{i,j}\middle| N(t), N_i\right]\middle| N(t)\right]\right] \\
&= \mathbf{E}\left[\sum_{i=0}^{N(t)} \mathbf{E}\left[N_i \mathbf{E}\left[I_{i,j}\middle| N(t)\right]\middle| N(t)\right]\right] \quad (I_{i,j} \text{ are independent of } N_i) \\
&= \mathbf{E}\left[\sum_{i=0}^{N(t)} p\mathbf{E}\left[N_i\middle| N(t)\right]\right] \\
&= \mathbf{E}\left[\sum_{i=0}^{N(t)} p\mathbf{E}\left[N_i\right]\right] \quad (N(t) \text{ is independent of } N_i) \\
&= p\mathbf{E}\left[N_i\right]\mathbf{E}\left[N(t)\right] \\
&= \lambda t p \mathbf{E}\left[N_i\right] \\
&= \lambda \sum_{j=1}^{\infty} j\alpha_j \int_0^t G(s) ds \ .
\end{aligned}
$$

Problem 2.20 The key thing here is to apply the conditional distribution of arrival times given a Poisson number of events in an interval. Under the conditioning, the unordered arrival times are distributed as IID uniform random variables. We apply this representation in the first step.

Assume that $n = \sum_{i=1}^{k} n_i$. Note that given $N(t) = n$ the unordered set of arrival times are independent uniform $(0, t)$. The probability that an arbitrary event is type i is thus

$$
p_i \equiv \frac{1}{t} \int_0^t P_i(x) dx \ .
$$

$$
\begin{aligned}
\mathbf{P}(N_i(t) = n_i, i = 1, \cdots, k) &= \mathbf{P}(N_i(t) = n_i, i = 1, \cdots, k | N(t) = n) \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \frac{n!}{n_1! \cdots n_k!} p_1^{n_1} \cdots p_k^{n_k} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad \text{(multinomial)} \\
&= \frac{e^{-\lambda p_1 t}(\lambda p_1 t)^{n_1}}{n_1!} \cdots \frac{e^{-\lambda p_k t}(\lambda p_k t)^{n_k}}{n_k!} \ .
\end{aligned}
$$

Hence we're done.(Why?)

Problem 2.21 The key idea here is to apply Problem 2.20. In this problem, the state of an individual varies, which might cause trouble to put into the setting of Problem 2.20. But, if we fix the observation time t, then the probability that an individual who arrived at time s is in state i at time t is just $\alpha_i(t - s)$ and if we define $P_i(s) \equiv \alpha_i(t - s)$, then we can apply

Problem 2.20. Furthermore,

$$\begin{aligned}\mathbf{E}[N_i(t)] &= \lambda \int_0^t P_i(x)dx \\ &= \lambda \int_0^t \alpha_i(t-x)dx \\ &= \lambda \int_0^t \alpha_i(y)dy \\ &= \lambda \int_0^t \mathbf{E}[I_i(y)]dy \\ &= \lambda \mathbf{E}\left[\int_0^t I_i(y)dy\right]\end{aligned}$$

which leads to the desired interpretation of the mean.

Now, are we done? Unless you are an advanced reader, not yet. The problem doesn't mention on the number of states i. Hence it is possible that there are countably many states and the random variables, $N_i(t)$, are countably many. Then what is the definition of independence among countably many random variables. One general fact in mathematics and probability is that when we define a property among countably many objects, we require that the property holds among any finitely many ones. (Recall the definition of basis in infinite dimensional vector space.) So, to prove the independence of $N_i(t)$ it is sufficient to show it under finite states.