IEOR 6711: Stochastic Models I, Professor Whitt

Solutions to Homework Assignment 5, not to be turned in.

Problem 2.25 Just as in Problem 2.20, the key thing here is to apply the conditional distribution of arrival times given a Poisson number of events in an interval. Under the conditioning, the unordered arrival times are distributed as IID uniform random variables. We apply this representation.

Let X be a contribution by an event occurred at random time distributed by uniform$(0, t)$. If we denote the occurrence time of the event by U (which is uniform $(0, t)$), then

$$\begin{aligned}
\mathbf{P}(X \le x) &= \mathbf{E}[\mathbf{P}(X \le x|U)] \\
&= \mathbf{E}[F_U(x)] \\
&= \int_0^t F_u(x) \frac{1}{t} du \\
&= \frac{1}{t} \int_0^t F_u(x) du \ .
\end{aligned}$$

Hence, if we define that X_T is a contribution of an event occurred at random time T and $U_1, \cdots, U_k, \cdots$ are random samples from uniform$(0, t)$, then $X \stackrel{d}{=} X_{U_i}$ and

$$\begin{aligned}
W &\equiv \sum_{i=1}^{N(t)} X_{U_i} \\
&= \sum_{i=1}^{N} X_{U_i} \\
&= \sum_{i=1}^{N} X_i
\end{aligned}$$

where N is a Poisson random variable with mean λt and X_i are IID with distribution given above.

Problem 2.30 (a) No. It should be intuitively clear that the answer is indeed no, but providing a proof is not so easy. It is not hard to make a proof in special cases. For example, when $\lambda(t)$ is a monotonically (strictly) decreasing function. Then the larger T_1, the larger T_2. However, the variables are *never* independent if the arrival process is nonhomogeneous. That is harder to prove. We will not really try. We will be impressed if you work that out.

(b) No. In the special case above, T_1 should be larger than T_2.

(c)

$$\begin{aligned}
\mathbf{P}(T_1 \ge t) &= \mathbf{P}(N(t) = 0) \\
&= e^{-\int_0^t \lambda(s) ds}
\end{aligned}$$

(d)

$$
\begin{aligned}
\mathbf{P}(T_2 \geq t) &= \mathbf{E}[\mathbf{P}(T_2 \geq t | T_1)] \\
&= \int_0^\infty \mathbf{P}(T_2 \geq t | T_1 = s) f_{T_1}(s) ds \\
&= \int_0^\infty e^{-\int_s^{t+s} \lambda(u)du} \lambda(s) e^{-\int_0^s \lambda(u)du} ds \\
&= \int_0^\infty e^{-\int_0^{t+s} \lambda(u)du} \lambda(s) ds \\
&= \int_0^\infty e^{-m(t+s)} \lambda(s) ds \ .
\end{aligned}
$$

Problem 2.31 A key initial step is to observe that the mean function $m(t)$ is a strictly increasing continuous function, by virtue of the assumptions. (The assumption $\lambda(\cdot) > 0$ is important.) Then m has a unique well defined inverse m^{-1}, which itself has a unique well defined inverse m.

For the rest of the way, we check the parts of Definition 2.1.1 one by one.

(i) $N^*(0) = N(m^{-1}(0)) = N(0) = 0$.

(ii) For $t > 0$ and $s > 0$, $m^{-1}(t+s) > m^{-1}(t) > 0$. $N^*(t+s) - N^*(t) = N(m^{-1}(t+s)) - N(m^{-1}(t))$ and $N(m^{-1}(t)) = N^*(t)$ are independent since $(m^{-1}(t), m^{-1}(t+s)]$ and $[0, m^{-1}(t)]$ are non-overlapping. So $N^*(\cdot)$ increases independently.

(iii) For $s, t \geq 0$,

$$
\begin{aligned}
& \mathbf{P}(N^*(t+s) - N^*(s) = n) \\
= \ & \mathbf{P}(N(m^{-1}(t+s)) - N(m^{-1}(s)) = n)) \\
= \ & e^{-m(m^{-1}(t+s)) + m(m^{-1}(s))} \frac{[m(m^{-1}(t+s)) - m(m^{-1}(s))]^n}{n!} \\
= \ & \frac{e^{-t} t^n}{n!}
\end{aligned}
$$

Problem 2.32 (a) The easy way to proceed is to apply Problem 2.31 above. To do so, we need to assume that $m(t)$ is continuous and strictly increasing. So assume that is the case. Then, from Problem 2.31, $N^*(m(t)) = N(t)$ and $N^*(\cdot)$ is a Poisson process with rate 1. Hence if we set the unordered arrival times of $N(\cdot)$ by $V_1, \cdots, V_n$, then $U_1 = m(V_1), \cdots, U_n = m(V_n)$ are those of $N^*(\cdot)$. But we know that the unordered arrival times $U_1, \cdots, U_n$ of $N^*(\cdot)$ given that $N^*(m(t)) = n$ are random samples from uniform$(0, m(t))$.

$$
\begin{aligned}
\mathbf{P}(V_i \leq x) &= \mathbf{P}(m(V_i) \leq m(x)) \\
&= \mathbf{P}(U_i \leq m(x)) \\
&= \frac{\min\{m(x), m(t)\}}{m(t)} \ .
\end{aligned}
$$

We can treat the general case of m as the limit of strictly increasing m. We thus obtain the formula above, by taking the limit.

(b) Note this question is exactly an $M_t/G/\infty$ queue question, so we can apply Theorem 1 in the "Physics" paper. The number of workers out has a Poisson distribution with the mean given by the mean formula $Mean(t) \equiv p = \int_0^t \lambda(x)\bar{F}(t-x)\,dx$. The Poisson distribution property implies that the variance equals the mean.

Below we give a direct derivation: Let $N(t)$ denote the number of accidents by time t. Let I be the indicator representing that an injured person is out of work at time t. Let V be the time of accident.

$$\begin{aligned}\mathbf{P}(I=1|N(t)) &= \mathbf{E}[\mathbf{P}(I=1|N(t),V)|N(t)] \\ &= \int_0^t \bar{F}(t-x)\frac{dm(x)}{m(t)} \\ &= \frac{1}{m(t)}\int_0^t \bar{F}(t-x)\lambda(x)dx \\ &= \frac{p}{m(t)}\ .\end{aligned}$$

Now $X(t) = \sum_{i=1}^{N(t)} I_i$ and $X(t)|N(t)$ is a binomial$(N(t), \frac{p}{m(t)})$. Hence

$$\mathbf{E}[X(t)|N(t)] = N(t)\frac{p}{m(t)}$$

and

$$\mathbf{V}ar[X(t)|N(t)] = N(t)\frac{p}{m(t)}\left(1-\frac{p}{m(t)}\right)\ .$$

Therefore

$$\mathbf{E}(X(t)) = \mathbf{E}[N(t)]\frac{p}{m(t)} = p$$

and

$$\begin{aligned}\mathbf{V}ar[X(t)] &= \mathbf{V}ar(\mathbf{E}[X(t)|N(t)]) + \mathbf{E}[\mathbf{V}ar(X(t)|N(t))] \\ &= \mathbf{V}ar(N(t))\left(\frac{p}{m(t)}\right)^2 + \mathbf{E}[N(t)]\frac{p}{m(t)}\left(1-\frac{p}{m(t)}\right) \\ &= \frac{p^2}{m(t)} + p\left(1-\frac{p}{m(t)}\right) \\ &= p\ .\end{aligned}$$

Problem 2.33 Here we have a Poisson random measure problem.

Let the reference point be x and $B(x,r)$ represent the circular region of radius r with center at x.

(a)

$$\begin{aligned}\mathbf{P}(X > t) &= \mathbf{P}(\text{no event in } B(x,t)) \\ &= e^{-\lambda\pi t^2} .\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{E}[X] &= \int_0^\infty \mathbf{P}(X > t)dt \\ &= \int_0^\infty e^{-\lambda\pi t^2} dt \\ &= \sqrt{\frac{1}{\lambda}} \int_0^\infty \frac{1}{\sqrt{2\pi\frac{1}{2\lambda\pi}}} e^{-\lambda\pi t^2} dt \\ &= \sqrt{\frac{1}{\lambda}} \times \frac{1}{2} \\ &= \frac{1}{2\sqrt{\lambda}} .\end{aligned}$$

(c) Since $\pi R_i^2 - \pi R_{i-1}^2$ is the area of the region between the circle of radius R_i and the one of radius of R_{i-1}, we have

$$\begin{aligned}\mathbf{P}(\pi R_i^2 - \pi R_{i-1}^2 > a | R_j, j < i) &= \mathbf{P}(\text{no event in the area a}) \\ &= e^{-\lambda a} .\end{aligned}$$

Problem 2.35 **(a)** No, because knowing the number of events in some interval is informative about the value of τ, which gives information about the distribution of the number of events in a non-overlapping interval.

(b) Yes, because the occurrences in any non-overlapping intervals are independent.

Problem 2.39 $s < t$. Let $X(t) = \sum_{i=1}^{N(t)} Y_i$. First note that $X(t)$ possesses stationary independent increments.

$$\begin{aligned}\mathbf{E}[X(s)X(t)] &= \mathbf{E}[X(s)(X(t) - X(s) + X(s))] \\ &= \mathbf{E}[X(s)(X(t) - X(s)) + \mathbf{E}[X(s)^2] \\ &= \mathbf{E}[X(s)]\mathbf{E}[X(t-s)] + \mathbf{E}[X(s)^2] \\ &= \lambda s\mathbf{E}[Y]\lambda(t-s)\mathbf{E}[Y] + \lambda s E[Y^2] + (\lambda s\mathbf{E}[Y])2 \\ &= \lambda^2 st\mathbf{E}[Y]^2 + \lambda s E[Y^2] .\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{C}ov(X(s), X(t)) &= \mathbf{E}[X(s)X(t)] - \mathbf{E}[X(s)]\mathbf{E}[X(t)] \\ &= \lambda^2 st\mathbf{E}[Y]^2 + \lambda s E[Y^2] - \lambda s\mathbf{E}[Y]\lambda t\mathbf{E}[Y] \\ &= \lambda s E[Y^2] .\end{aligned}$$