IEOR 6711: Stochastic Models I, Professor Whitt

SOLUTIONS to Homework Assignment 8.

Problem 3.19

$$
\begin{aligned}
P\{S_{N(t)} \le s\} &= \sum_{n=0}^{\infty} P\{S_n \le s, S_{n+1} > t\} \\
&= \bar{G}(t) + \sum_{n=1}^{\infty} P\{S_n \le s, S_{n+1} > t\} \\
&= \bar{G}(t) + \sum_{n=1}^{\infty} \int_0^{\infty} P\{S_n \le s, S_{n+1} > t | S_n = y\} d(G * F_{n-1})(s) \\
&= \bar{G}(t) + \int_0^s \bar{F}(t-y) dm_D(y)
\end{aligned}
$$

Problem 3.20

(a) Say that a renewal occurs when that pattern appears. By Blackwell's theorem for renewal processes we obtain

$$E[time] = \frac{1}{(1/2)^7} = 2^7$$

(b) By Blackwell's theorem

E[time between HHTT renewals] $= E$[time between HTHT renewals] $= 16$

But the HHTT renewal process is an ordinary one and so the mean time until HHTT occurs is 16 whereas the HTHT process is a delayed renewal process and so the mean time until HTHT occurs is greater than 16.

Problem 3.29

Let L denote the lifetime of a car with distribution function $F(\cdot)$.

(a) Under the policy of replacements at A,

$$\text{Cost of cycle} = \begin{cases} C_1 + C_2 & \text{if } L \le A \\ C_1 - R(A) & \text{if } L > A \end{cases}$$

and

$$\text{Length of cycle} = \begin{cases} L & \text{if } L \le A \\ A & \text{if } L > A \end{cases} .$$

Then

$$\frac{\mathsf{E}[\text{Cost}]}{\mathsf{E}[\text{Time}]} = \frac{C_1 + C_2 F(A) - R(A)\bar{F}(A)}{\int_0^A x dF(x) + A\bar{F}(A)}$$

(Validate the final formula by yourself. If you are confusing, utilize the *indicator* to combine the *if*-clauses into one function as I said in the first recitation.)

(b) Condition on the life of the initial car.

$$\begin{aligned}
\mathsf{E}[\text{Length of cycle}] &= \int_0^\infty \mathsf{E}[\text{Length}|L=x]dF(x) \\
&= \int_0^A x dF(x) + \int_A^\infty (A + \mathsf{E}[\text{Length}])dF(x) \\
&= \int_0^A x dF(x) + (A + \mathsf{E}[\text{Length}])\bar{F}(A) \\
&= \frac{\int_0^A x dF(x) + A\bar{F}(A)}{F(A)}
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathsf{E}[\text{Cost of cycle}] &= \int_0^\infty \mathsf{E}[\text{Cost}|L=x]dF(x) \\
&= \int_0^A (C_1 + C_2)dF(x) + (C_1 - R(A) + \mathsf{E}[Cost])\bar{F}(A) \\
&= \frac{C_1 + C_2 F(A) - R(A)\bar{F}(A)}{F(A)}\ .
\end{aligned}$$

Then

$$\frac{\mathsf{E}[\text{Cost}]}{\mathsf{E}[\text{Time}]} = \text{same as in (a).}$$

Problem 3.31

Let μ_i and ν_i denote the means of F_i and G_i, respectively for $i = 1, 2, 3, 4$. Then,

$$\lim P\{\text{i is working at t}\} = \mu_i/(\mu_i + \nu_i), i = 1, 2, 3, 4$$

Now, if p_i is the probability that component i is working, then

$$P\{\text{system works}\} = (p_1 + p_2 - p_1 p_2)(p_3 + p_4 - p_3 p_4)$$

Hence $\lim P\{\text{system works at t}\}$ is equal to the preceding expression with $p_i = \mu_i/(\mu_i + \nu_i), i = 1, 2, 3, 4$

Problem 3.32

(a) $1 - P_0$ = average number in service $= \lambda\mu$

(b) By alternating renewal processes

$$P_0 = \text{proportion of time empty} = \frac{E[I]}{E[I] + E[B]}$$

where I is an idle period and B a busy period. But clearly I is exponential with rate λ and so

$$1 - \lambda\mu = \frac{1/\lambda}{1/\lambda + E[B]} \text{ or } E[B] = \frac{\mu}{1 - \lambda\mu}$$

(c) Let C denote the number of customers served in a busy period B and let S_i denote the service time of the i-th customer, $i \geq 1$. Then

$$B = \sum_{i=1}^{C} S_i$$

and by Wald's equation

$$E[C] = \frac{E[B]}{E[S]} = \frac{1}{1 - \lambda\mu}$$