# IEOR 6711: Stochastic Models I

## Solutions to Homework Assignment 9

**Problem 4.1** Let $D_n$ be the random demand of time period $n$. Clearly $D_n$ is i.i.d. and independent of all $X_k$ for $k < n$. Then we can represent $X_n + 1$ by

$$X_{n+1} = \max\{0, X_n \cdot \mathbf{1}_{[s,\infty)}(X_n) + S \cdot \mathbf{1}_{[0,s)}(X_n) - D_{n+1}\}$$

which depends only on $X_n$ since $D_{n+1}$ is independent of all history. Hence $\{X_n, n \geq 1\}$ is a Markov chain. It is easy to see assuming $\alpha_k = 0$ for $k < 0$,

$$P_{ij} = \begin{cases} \alpha_{S-j} & \text{if } i < s, j > 0 \\ \sum_{k=S}^{\infty} \alpha_k & \text{if } i < s, j = 0 \\ \alpha_{i-j} & \text{if } i \geq s, j > 0 \\ \sum_{k=i}^{\infty} \alpha_k & \text{if } i \geq s, j = 0 \end{cases}$$

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The following three problems (4.2, 4.4, 4.5) needs a fact:

$$\mathsf{P}(A \cap B|C) = \mathsf{P}(A|B \cap C)\mathsf{P}(B|C)$$

which requires a proof to use. Try to prove it by yourself.

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**Problem 4.2** Let $\mathcal{S}$ be the state space. First we show that

$$\mathsf{P}(X_{n_k+1} = j|X_{n_1} = i_1, \cdots, X_{n_k} = i_k) \quad = \quad \mathsf{P}(X_{n_k+1} = j|X_{n_k} = i_k)$$

by the following : Let $A = \{X_{n_k+1} = j\}$, $B = \{X_{n_1} = i_1, \cdots, X_{n_k} = i_k\}$ and $B_b, b \in \mathcal{I}$ are elements of $\{(X_l, l \leq n_k, l \neq n_1, \cdots, l \neq n_k) : X_l \in \mathcal{S}\}$.

$$\begin{aligned} \mathsf{P}(A|B) &= \sum_{b\in\mathcal{I}} \mathsf{P}(A \cap B_b|B) \\ &= \sum_{b\in\mathcal{I}} \mathsf{P}(A|B_b \cap B)\mathsf{P}(B_b|B) \\ &= \sum_{b\in\mathcal{I}} \mathsf{P}(A|X_{n_k} = i_k)\mathsf{P}(B_b|B) \\ &= \mathsf{P}(A|X_{n_k} = i_k) \sum_{b\in\mathcal{I}} \mathsf{P}(B_b|B) \\ &= \mathsf{P}(A|X_{n_k} = i_k)\mathsf{P}(\Omega|B) \\ &= \mathsf{P}(X_{n_k+1} = j|X_{n_k} = i_k) \ . \end{aligned}$$

We consider the mathematical induction on $l \equiv n - m$. For $l = 1$, we just showed. Now assume that the statement is true for all $l \le l^*$ and consider $l = l^* + 1$:

$$\begin{aligned}
& \mathsf{P}(X_n = j | X_{n_1} = i_1, \cdots, X_{n_k} = i_k) \\
= & \sum_{i \in \mathcal{S}} \mathsf{P}(X_n = j, X_{n-1} = i | X_{n_1} = i_1, \cdots, X_{n_k} = i_k) \\
= & \sum_{i \in \mathcal{S}} \mathsf{P}(X_n = j | X_{n-1} = i, X_{n_1} = i_1, \cdots, X_{n_k} = i_k) \mathsf{P}(X_{n-1} = i | X_{n_1} = i_1, \cdots, X_{n_k} = i_k) \\
= & \sum_{i \in \mathcal{S}} \mathsf{P}(X_n = j | X_{n-1} = i) \mathsf{P}(X_{n-1} = i | X_{n_k} = i_k) \quad \text{By } l \le l^* \text{ cases} \\
= & \sum_{i \in \mathcal{S}} \mathsf{P}(X_n = j | X_{n-1} = i, X_{n_k} = i_k) \mathsf{P}(X_{n-1} = i | X_{n_k} = i_k) \\
= & \sum_{i \in \mathcal{S}} \mathsf{P}(X_n = j, X_{n-1} = i | X_{n_k} = i_k) \\
= & \mathsf{P}(X_n = j | X_{n_k} = i_k)
\end{aligned}$$

which completes the proof for $l = l^* + 1$ case.

**Problem 4.3** Simply by *Pigeon hole principle* which saying that if $n$ pigeons return to their $m(< n)$ home (through hole), then at least one home contains more than one pigeon. Consider any path of states $i_0 = i, i_1, \cdots, i_n = j$ such that $P_{i_k, i_{k+1}} > 0$. Call this a path from $i$ to $j$. If $j$ can be reached from $i$, then there must be a path from $i$ to $j$. Let $i_0. \cdots, i_n$ be such a path. If all of values $i_0, \cdots, i_n$ are not distinct, then there must be a subpath from $i$ to $j$ having fewer elements (for instance, if $i, 1, 2, 4, 1, 3, j$ is a path, then so is $i, 1, 3, j$). Hence, if a path exists, there must be one with all distinct states.

**Problem 4.4** Let $Y$ be the first passage time to the state $j$ starting the state $i$ at time 0.

$$\begin{aligned}
P_{ij}^n &= \mathsf{P}(X_n = j | X_0 = i) \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j, Y = k | X_0 = i) \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j | Y = k, X_0 = i) \mathsf{P}(Y = k | X_0 = i) \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j | X_k = j) \mathsf{P}(Y = k | X_0 = i) \\
&= \sum_{k=0}^{n} P_{jj}^{n-k} f_{ij}^k
\end{aligned}$$

**Problem 4.5** **(a)** The probability that the chain, starting in state $i$, will be in state $j$ at time $n$ without ever having made a transition into state $k$.

**(b)** Let $Y$ be the last time leaving the state $i$ before first reaching to the state $j$ starting the state $i$ at time 0.

$$\begin{aligned}
P_{ij}^n &= \mathsf{P}(X_n = j | X_0 = i) \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j, Y = k | X_0 = i) \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j, Y = k, X_k = i | X_0 = i) \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j, Y = k | X_k = i, X_0 = i)\mathsf{P}(X_k = i | X_0 = i) \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j, Y = k | X_k = i) P_{ii}^k \\
&= \sum_{k=0}^{n} \mathsf{P}(X_n = j, X_l \neq i, l = k+1, \cdots, n-1 | X_k = i) P_{ii}^k \\
&= \sum_{k=0}^{n} P_{ij/i}^{n-k} P_{ii}^k
\end{aligned}$$

**Problem 4.7**

**(a)** $\infty$

Here is an argument: Let $x$ be the expected number of steps required to return to the initial state (the origin). Let $y$ be the expected number of steps to move to the left 2 steps, which is the same as the expected number of steps required to move to the right 2 steps. Note that the expected number of steps required to go to the left 4 steps is clearly $2y$, because you first need to go to the left 2 steps, and from there you need to go to the left 2 steps again. Then, consider what happens in successive pairs of steps: Using symmetry, we get

$$x = 2 + (0 \times (1/2) + y \times (1/2) = 2 + y/2$$

and

$$y = 2 + (0 \times (1/4) + y \times (1/2) + (2 * y) \times (1/4)$$

If we subtract $y$ from both sides, this last equation yields

$$2 = 0 \ .$$

Hence there is no finite solution. The quantity $y$ must be infinite; a finite value cannot solve the equation.

**(b)** Note that the expected number of returns in $2n$ steps is the sum of the probabilities of returning in $2k$ steps for $k$ from 1 to $n$, each term of which is binomial. Thus, we have

$$E[N_{2n}] = \sum_{k=1}^{n} \frac{(2k)!}{k!k!}(1/2)^{2k} \ ,$$

which can be shown to be equal to the given expression by mathematical induction.

**(c)** We say that $f(n) \sim g(n)$ as $n \to \infty$ if

$$f(n)/g(n) \to 1 \quad \text{as} \quad n \to \infty \ .$$

By Stirling's approximation,

$$(2n+1)\frac{(2n)!}{n!n!}(1/2)^2 n \sim 2\sqrt{n/\pi} \ ,$$

so that

$$E[N_n] \sim \sqrt{2n/\pi} \quad \text{as} \quad n \to \infty \ .$$

**Problem 4.8 (a)**

$$P_{ij} = \frac{\alpha_j}{\sum_{k=i+1}^{\infty} \alpha_k} \ , \quad j > i$$

**(b)** $\{T_i, i \geq 1\}$ is not a Markov chain - the distribution of $T_i$ does depend on $R_i$. $\{(R_{i+1}, T_i), i \geq 1\}$ is a Markov chain.

$$\begin{aligned} \mathsf{P}(R_{i+1} = j, T_i = n | R_i = l, T_{i-1} = m) &= \frac{\alpha_j}{\sum_{k=l+1}^{\infty} \alpha_k} \left( \sum_{k=0}^{l} \alpha_k \right)^{n-1} \sum_{k=l+1}^{\infty} \alpha_k \\ &= \alpha_j \left( \sum_{k=0}^{l} \alpha_k \right)^{n-1} \ , \quad j > l \end{aligned}$$

**(c)** If $S_n = j$ then the $(n+1)^{st}$ record occurred at time $j$. However, knowledge of when these $n+1$ records occurred does not yield any information about the set of values $\{X_1, \cdots, X_j\}$. Hence, the probability that the next record occurs at time $k$, $k > j$, is the probability that both $\max\{X_1, \cdots, X_j\} = \max\{X_1, \cdots, X_{k-1}\}$ and that $X_k = \max\{X_1, \cdots, X_k\}$. Therefore, we see that $\{S_n\}$ is a Markov chain with

$$P_{jk} = \frac{j}{k-1} \, \frac{1}{k} \ , \quad k > j \ .$$

**Problem 4.11 (a)**

$$\begin{aligned} \sum_{n=1}^{\infty} P_{ij}^n &= \mathsf{E}[\text{number of visits to } j | X_0 = i] \\ &= \mathsf{E}[\text{number of visits to } j | \text{ever visit } j, X_0 = i] f_{ij} \\ &= (1 + \mathsf{E}[\text{number of visits to } j | X_0 = j]) f_{ij} \\ &= \frac{f_{ij}}{1 - f_{jj}} < \infty \ . \end{aligned}$$

since 1 + number of visits to $j | X_0 = j$ is geometric with mean $\frac{1}{1-f_{jj}}$.

**(b)** Follows from above since

$$\begin{aligned}\frac{1}{1-f_{jj}} &= 1+\mathsf{E}[\text{number of visits to } j|X_0=j] \\ &= 1+\sum_{n=1}^{\infty} P_{jj}^{n}\ .\end{aligned}$$

**Problem 4.12** If we add the irreducibility of $\mathbf{P}$, it is easy to see that $\boldsymbol{\pi} = \frac{1}{n}\mathbf{1}$ is a (and the unique) limiting probability.