

## IEOR 6711: Professor Whitt

### Notes on Laplace Transforms and Their Inversion

“The shortest path between two truths in the real domain passes through the complex domain;” Jacques Hadamard (1865-1963).

#### 1. Basic Definition

Let  $X$  be a nonnegative real-valued random variable with probability density function (pdf)  $f \equiv f_X$ . Then the Laplace transform of the random variable  $X$ , and also the Laplace transform of the pdf  $f$ , is

$$E[e^{-sX}] \equiv \hat{f}(s) \equiv \int_0^{\infty} e^{-st} f(t) dt, \quad (1)$$

where  $s$  is a complex variable with nonnegative real part. (If we write  $s = u + vi$ , where  $i \equiv \sqrt{-1}$  and  $u$  and  $v$  are real numbers, then  $u$  is  $Re(s)$  (the real part of  $s$ ) and  $v$  is  $Im(s)$  (the imaginary part of  $s$ ). Google it if this is new to you. We assume that  $u \geq 0$ , which guarantees that the integral is well defined provided that the function  $f$  is itself integrable.)

If we replaced  $-s$  by  $+s$ , where we regard  $s$  as real, we would have the moment generating function (mgf), discussed in Ross; see Section 1.4. A problem with the mgf is that it is not always finite. For example if  $f(t) = A/(1+t)^p$ , for  $t \geq 0$ , where  $A > 0$  and  $p > 1$  are parameters, then the mgf is infinite.

But the Laplace transform is well defined for any nonnegative random variable (probability distribution on the positive halfline  $[0, \infty)$ ).

#### 2. Project on Numerical Inversion

You are asked to do a short project on numerical inversion. Associated with that project is a numerical homework. It concerns learning about a method to numerically invert Laplace transforms. The idea is to write your own code, presumably using MATLAB, but you could use another language, C++, say. You will need to work with complex numbers. There is a specific homework assignment for this optional assignment, which involves solving some problems with your code. For this, you can follow the papers on the computational tools web page. In particular, you should look at the 1995 JoC paper.

#### 3. Project Management

We now illustrate how numerical inversion can be applied. We discuss Example 1.1.1 on pages 3-5 of the book chapter by Abate, Choudhury and Whitt (1999), referred to here as ACW, given on the tools web page. Suppose that we are managing a project, consisting of four steps. We want to know the probability distribution of the time  $T$  to complete the entire project. Let  $X_i$  be the time required to complete step  $i$  of the project,  $i = 1, \dots, 4$ . Assume that the total time can be represented as the sum of the four separate steps, i.e.,

$$T = X_1 + X_2 + X_3 + X_4. \quad (2)$$

We are thus assuming one task is done after the other. We have to finish task 1 before we start task 2, and so forth. Moreover, suppose that the four step times  $X_i$  are **mutually independent random variables**, with known pdf's  $f_i$ . We want to compute the cdf of  $T$ ; i.e., we want to calculate

$$G(t) \equiv P(T \leq t), \quad t \geq 0. \quad (3)$$

Directly, the cdf  $G$  can be expressed as a relatively complicated convolution integral, as shown in (1.1) of ACW. That single integral extends to a multiple (three-dimensional) integral when we consider the sum of 4 random variables or, equivalently, the convolution of 4 distributions.

An attractive alternative approach is to exploit Laplace transforms. We can write

$$\hat{g}(s) = \hat{f}_1(s) \times \hat{f}_2(s) \times \hat{f}_3(s) \times \hat{f}_4(s). \quad (4)$$

where  $\hat{f}_i(s)$  is the Laplace transform of the pdf  $f_i$ , as defined above in (1) and

$$\hat{g}(s) = E[e^{-sT}] = \int_0^\infty e^{-st} g(t) dt, \quad (5)$$

where  $g$  is the pdf of  $T$ . To see that is so, note that

$$E[e^{-s(X_1+\dots+X_4)}] = E[e^{-s(X_1)}e^{-s(X_2)}e^{-s(X_3)}e^{-s(X_4)}] = \hat{f}_1(s) \times \hat{f}_2(s) \times \hat{f}_3(s) \times \hat{f}_4(s). \quad (6)$$

We use independence to have the expectation of the product be equal to the product of the expectations; see p. 8 of Ross. You want more than is given there. You want to know that  $E[h_1(X_1)h_2(X_2)] = E[h_1(X_1)]E[h_2(X_2)]$  for all functions  $h_1$  and  $h_2$  if  $X_1$  and  $X_2$  are independent.

The above shows that we can calculate the Laplace transform of  $T$ , denoted by  $\hat{g}(s)$ , simply as the product of the Laplace transforms of  $X_i$ . Since the cdf  $G$  is the integral of the pdf  $g$ , the Laplace transforms are related by

$$\hat{G}(s) \equiv \int_0^\infty e^{-st} G(t) dt = \frac{\hat{g}(s)}{s}. \quad (7)$$

(The last formula is a standard formula for Laplace transforms, shown by using integration by parts.) Hence, we can calculate  $G(t)$  for any  $t$  of interest by numerically inverting its Laplace transform,  $\hat{G}(s) = \hat{g}(s)/s$ . That can be done quite quickly with an inversion program. See the papers on line for discussion.

Sometimes we are not given the component distributions or transforms, but instead are given only estimates of the first two moments. The mean is easy:

$$E[T] = E[X_1] + E[X_2] + E[X_3] + E[X_4].$$

Since the random variables are independent, the variances add as well:

$$Var[T] = Var[X_1] + Var[X_2] + Var[X_3] + Var[X_4].$$

But that does not give us the distribution of  $T$ . We could fit a distribution to the first two moments (or mean and variance) of  $T$ , given the relations above, but it is often better to fit appropriate distributions to the individual  $X_i$ , and then calculate their transforms, and perform the inversion above. For example, we might fit the distribution of  $X_i$  to a gamma distribution. The transform of a gamma distribution is

$$\hat{f}(s) = \left( \frac{\lambda}{\lambda + s} \right)^\nu,$$

where  $\lambda$  and  $\nu$  are positive parameters. The mean is  $\nu/\lambda$  and the variance is  $\nu/\lambda^2$ . We can thus easily fit the parameters  $\nu$  and  $\lambda$  to the mean and variance. We then can carry out the inversion described above.

We close by discussing inversion, giving reference to the 1995 JoC paper.

#### 4. A brief recap of the Euler algorithm, with reference to the 1995 JoC paper.

One way to start is with the **Bromwich contour integral**. It gives the desired function value  $f(t)$  in terms of the transform  $\hat{f}(s)$ , see (2) on page 37. In (2) we show that the contour integral is easily represented as an integral of a real-valued function of a real variable over the positive half line  $[0, \infty)$ . That real integral is given in the last line of (2). One tricky step in getting there is that we must use that the inversion integral yields the value 0 for  $t < 0$ . That is part of the Bromwich theorem. That is used in the last step in (2), as well as basic properties of even and odd functions. The steps in (2) are better explained in the 1999 survey paper on page 7.

From (2), it suffices to perform the numerical integration, also called numerical quadrature. There are two steps, discretizing to get a series and truncating the infinite series to get a finite sum. In the end we use a convergence-improvement technique instead of simple sum. That produces a finite weighted sum. The numerical-integration technique we use is the **trapezoidal rule**. That is usually thought to be a primitive method, but it turns out to be effective in this context. The trapezoidal rule yields formula (4). A key step is to determine the **discretization error** in applying the trapezoidal rule with step size  $h$ . That is addressed by aliasing below. Given (4), we perform a change of variables to force all the cosine terms to be  $+1$ ,  $-1$  or  $0$ . The result is (5). We do that to get an alternating series (more precisely, a series that is likely to be eventually an alternating series), which is convenient for convergence improvement.

An alternative approach to the inversion is via Fourier series. As motivation, we can observe that (4) is in fact a trigonometric series; i.e., it is a Fourier series. We might thus ask what is the function that it is the Fourier series of? You can find notes on Fourier series from Mathworld. We obtain Fourier series expansions of functions on a finite interval or of functions which are periodic. That is, given a periodic function, we can construct its Fourier series. One approach is to determine the function for which (4) is its Fourier series.

However, we need not even know about the Bromwich integral. We could instead start with  $f$  and construct a periodic function and take its Fourier series. If done right, expression (4) will be the Fourier series of that periodic function. We can construct a periodic function of an arbitrary given function by **aliasing**, that is, by adding infinitely many translates of the function. Starting from a function  $g$ , the associated periodic function is  $g_p$  given in (6). However, we must do something to guarantee that the series converges. To get convergence, we first damp our original function by writing

$$g(t) = e^{-bt} f(t), \quad t \geq 0, \quad \text{for } b > 0.$$

Assuming that  $f$  is well behaved, e.g., if  $f$  is bounded, then the series in the definition of  $g_p$  will converge.

We observe that  $g_p$  is periodic. Then we construct the complex Fourier series of that periodic function, given in (7). A critical property is that the coefficients of the Fourier series can be expressed in terms of the transform values, which is precisely what we have to work with. That is shown in (8). The two steps together - aliasing and Fourier series - give us the **Poisson summation formula**, given in (9). That is a classic formula, with many many applications. We observe that  $f(t)$  itself is obtained by looking at the single term on the left in (9) corresponding to  $k = 0$ .

We then perform a change of variables, and obtain (10), which gives us once again formula (5), but with an explicit expression for the discretization error. We get that simple representation because  $f(t) = 0$  for  $t < 0$ . We then get a simple bound on the discretization error in (12), under the assumption that  $|f(t)| \leq 1$  for all  $t$ .

The remaining problem is to sum the infinite series in (5). We could do so by simple truncation, but it turns out to be much better to do a **convergence improvement technique**. Having made (5) a nearly alternating series, a natural candidate is Euler summation. We take the weighted average of the last 12 terms (with  $k$  ranging from 0 to  $m = 11$ ). Specifically, we take a binomial average of those last 12 terms, as shown in (15). I give you simple exercises in the homework, so that you can see the power of Euler summation. See Section 1.2.3 on p. 17-19 of the Summary paper for more on Euler summation.

In general, the algorithm is given in formulas (13)–(15). There are three parameters  $A$ ,  $n$  and  $m$ . We use  $A$  to control the discretization error, but we are not too greedy; we avoid roundoff error by not choosing  $A$  too large. (Roundoff error occurs in (13) because of the multiplication by  $e^{A/2}$ . Roundoff error is discussed on page 15 of the 1999 summary.) Reasonable values of the parameters are  $A = 19$ ,  $n = 38$  and  $m = 11$ . There is of course another parameter:  $t$ . That is the argument of the function we wish to compute: We are computing  $f(t)$ . The algorithm is summarized on page 16 of the 1999 summary article. Pseudo-code (actually UBASIC code) appears in the left-hand box on page 41 of the 1995 JoC paper.

## 5. Other Inversion Methods.

There are other inversion techniques. We may apply the Bromwich integral in a different way. For example, Talbot's method is based on trying to more carefully choose the contour of integration. But we do not even need to start with the Bromwich integral. We illustrate by briefly describing an entirely different inversion formula, involving differentiation instead of integration.

The **Post-Widder formula** is

$$f(t) = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} ((n+1)/t)^{n+1} \hat{f}^{(n)}((n+1)/t),$$

as given in Theorem 2 of the book chapter and (20) of the 1996 paper.

This formula can be derived and understood as

$$\frac{(-1)^n}{n!} ((n+1)/t)^{n+1} \hat{f}^{(n)}((n+1)/t) = E[f(X_{n,t})],$$

where  $X_{n,t}$  is a random variable with a gamma distribution having mean  $t$  and variance  $t^2/(n+1)$ . Since the variance goes to 0 as  $n \rightarrow \infty$ , we have

$$X_{n,t} \Rightarrow t \quad \text{as } n \rightarrow \infty.$$

Indeed, that can be proved from Chebyshev's inequality:

$$P(|X_{n,t} - t| > \epsilon) \leq \frac{\text{Var}(X_{n,t})}{\epsilon^2}.$$

Since  $\text{Var}(X_{n,t}) \rightarrow 0$  as  $n \rightarrow \infty$ , for any  $\epsilon > 0$ ,

$$P(|X_{n,t} - t| > \epsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Under extra regularity conditions (e.g., uniform integrability),

$$E[f(X_{n,t})] \rightarrow t \quad \text{as } n \rightarrow \infty .$$

To see that indeed

$$\frac{(-1)^n}{n!} ((n+1)/t)^{n+1} \hat{f}^{(n)}((n+1)/t) = E[f(X_{n,t})] ,$$

start by writing

$$(-1)^n \hat{f}^{(n)}(\lambda) = \int_0^\infty e^{-\lambda x} x^n f(x) dx .$$

Next let  $\lambda = (n+1)/t$  to get

$$(-1)^n \hat{f}^{(n)}((n+1)/t) = \int_0^\infty e^{-[(n+1)/t]x} x^n f(x) dx .$$

Then multiply both sides by  $((n+1)/t)^{n+1}$  and divide both sides by  $n!$  to get the desired equation.