A Concise Summary

Everything you need to know about exponential and Poisson

Exponential Distribution

Assume that $X \sim \exp(\lambda)$, by which we mean that X has an exponential distribution with *rate* λ. Then X has mean $1/\lambda$; i.e., $EX = 1/\lambda$. Also the variance is $Var(X) = (EX)^2 = 1/\lambda^2$. In addition, assume that $Y \sim \exp(\mu)$ and $X_i \sim \exp(\lambda_i)$ for $i = 1, \cdots, n$, where all these exponential random variables are independent.

1. *Lack of memory*: $P(X > s + t | X > s) = P(X > t)$ for all $s > 0$ and $t > 0$.
   (check the computation)
2. *Minimum*: $\min\{X, Y\} \sim \exp(\lambda + \mu)$
   (check the computation)
   and hence
   $\min\{X_1, \cdots, X_n\} \sim \exp(\lambda_1 + \cdots + \lambda_n)$ without computation.
3. *Maximum*: $X+Y = \min\{X, Y\}+\max\{X, Y\}$ tells us an easy way to compute $E[\max\{X, Y\}]$.)
4. *More on Minimum*: $P(X = \min\{X, Y\}) = P(X < Y) = \frac{\lambda}{\lambda+\mu}$: (check the computation)
   and hence
   $P(X_k = \min\{X_1, \cdots, X_n\}) = \frac{\lambda_k}{\lambda_1+\cdots+\lambda_n}$ without computation.
5. *Even more on Minimum*:The events $\{X = \min\{X, Y\}\}$ and $\{\min\{X, Y\} > t\}$ are independent for all t.
   and hence
   the events $\{X_k = \min\{X_1, \cdots, X_n\}\}$ and $\{\min\{X_1, \cdots, X_n\} > t\}$ are independent for all t.

Poisson Processes

Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ, referred to by $N(t)(\lambda)$. In addition, consider Poisson processes $N_j(t)(\lambda_j)$, $1 \leq j \leq m$.

1. *Interarrival Times*: The interarrival times of $N(t)(\lambda)$ are IID $exp(\lambda)$.
2. *Thinning (Type classification)* : When arrivals occur in the Poisson process $N(t)$, they are classified randomly and *independently* (the successive classifications are done independently according to the same probabilities) into classes (indexed by j) with probability $p_1, \cdots, p_m$. Let $N_j(t)$ be the input (arrival) process for class j (obtained with probability p_j). These newly created counting processes $N_j(t)$ are independent Poisson processes with rates λp_j.
3. *Superposition (Type aggregation)* : When *independent* Poisson arrivals $N_j(t)$ occur with rates λ_j, the total number of arrivals is a Poisson$(\lambda_1 + \cdots + \lambda_m)$.
4. *Conditioning* : When we know $N(t) = n$, the occurrence time of n arrivals are distributed as independent random variables, each uniform on the interval $[0, t]$.

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Poisson Process: A Special Case of Several Processes

It is useful to be aware that a Poisson process is a special case of several important stochastic processes. That leads to different equivalent definitions of a Poisson process, as in Definitions 2.1.1 and 2.1.2 of the Ross text. It also leads to different ways to analyze a Poisson process.

(a) **Poisson random measure**

Definition 2.1.1 can be viewed as a special case of a Poisson process defined as a Poisson random measure (on a subset of a Euclidean space $\mathbb{R}^k$ with an intensity function $\lambda(x)$, where $k = 1$ and the subset in $[0, \infty)$ and $\lambda(x) \equiv \lambda$ for some positive constant λ). A Poisson process (as well as a nonhogeneous Poisson process - Section 2.4 - can be viewed as a special case of a Poisson random measure. In the standard case, the underlying space is the positive real line $[0, \infty)$. But Poisson random measures can be defined on more general spaces, such as $\mathbb{R}^2$, corresponding to random points on the blackboard. Exercise 2.33 discusses a special case of a Poisson random measure in which the space is $\mathbb{R}^2$. We will exploit this perspective when we discuss the $M_t/GI/\infty$ infinite-server queue

(b) **CTMC**

Definition 2.1.2 can be viewed as a special case of a Poisson process defined as a a special case of a continuous-time Markov chain (CTMC). One way to characterize a CTMC is via its infinitesimal rate matrix, usually denoted by Q. For a Poisson process, we have $Q_{i,i+1} = \lambda$, $Q_{i,i} = -\lambda$ and $Q_{i,j} = 0$ for all other j. The rate matrix Q determines the probability transition matrix $P(t) \equiv (P_{i,j}(t))$ via a matrix ordinary differential equation (ODE)

$$\dot{P}(t) = P(t)Q = QP(t) \ ;$$

see the CTMC lecture notes posted on line for the end of the course in November (already posted). That corresponds to Definition 2.1.2. Notice that the "little oh" notation in (iii) of definition 2.1.2 just means that the function has a derivative (from the right at 0).

(c) **Lévy Process**

A Poisson Process can also be viewed a special case of a Lévy process. A Lévy process is a stochastic process with stationary and independent increments. Assume that it takes the value 0 at time 0. (This is another way to look at Definition 2.1.2.) The unique Lévy process with continuous sample paths is Brownian motion. (You do not need to directly assume that the increments have a normal distribution.) The unique Lévy process with sample paths having only unit jumps is the Poisson process. (You do not need to directly assume that the increments have a Poisson distribution.) The general Lévy process can be constructed from an independent Brownian motion and Poisson processes. This corresponds to Definition 2.1.1 (which is not stated in such a minimal elegant way).

(d) **renewal process**

A Poisson process is a special case of a renewal process (Chapter 3) in which the times between renewals have an exponential distribution. This corresponds to Proposition 2.2.1.

2. Basic Properties

Please pay attention to the basic properties on the Concise Summary page, such as Theorem 2.3.1 and Proposition 2.3.2.