IEOR 6711: Stochastic Models I

Fall 2013, Professor Whitt

Topic for Discussion, Thursday, October 3

Simulation of a Nonhomogeneous Poisson Process and Homework 5

1. Simulation by Thinning

You can apply time-varying thinning to simulate a nonhomogeneous Poisson process (NHPP), see pages 68-69 of Ross. Given an NHPP with time-varying rate function $\lambda(t)$, we choose a constant λ such that

$$\lambda(t) \le \lambda \quad \text{for all} \quad t, \quad 0 \le t \le T.$$

We then simulate a homogeneous Poisson process (HPP) with rate λ, e.g., by generating a sequence of i.i.d. exponential times between points, having mean $1/\lambda$. We look at the points of the HPP and assign some of them to be points of the NHPP with arrival rate function $\lambda(t)$. We let a point at time t in the HPP be also a point in the NHPP with probability $\lambda(t)/\lambda$, independent of the history up to time t, and we put no point otherwise. Thus the set of points of the NHPP so constructed is a subset of the points of the HPP. Each increment of the NHPP is necessarily less than or equal to the corresponding increment of the HPP with probability 1. The construction thus yields a bonafide NHPP with the desired rate. (You could check the properties in the definition.) However, this is a special construction.

2. Stochastic Order and Stochastic Comparison of NHPP's

If we have two NHPP's $N_1(t)$ and $N_2(t)$ with rate functions $\lambda_1(t)$ and $\lambda_2(t)$, respectively, then we can make strong stochastic order comparisons between the two NHPP's. For that, we use the notion of stochastic order; see §9.1 of the Ross textbook. Let $X \le_{st} Y$ denote stochastic order for real-valued random variables.

Definition 0.1 *Stochastic order $X \le_{st} Y$ holds if*

$$P(X > x) \equiv 1 - F_X(x) \le 1 - F_Y(x) \equiv P(Y > x) \quad \textit{for all} \quad x. \tag{1}$$

Of course, property (1) is equivalent to

$$F_X(x) \ge F_Y(x) \quad \text{for all} \quad x.$$

It turns out that another equivalent property is

$$E[h(X) \le E[h(Y)] \tag{2}$$

for all nondecreasing integrable real-valued functions h; see Proposition 9.1.2 of Ross.

We can conclude that

$$N_1(t_2) - N_1(t_1) \le_{st} N_2(t_2) - N_2(t_1)$$

for all $0 < t_1 < t_2$, because we can do a w.p.1 construction as above so that versions are ordered w.p.1, as described above. We construct both NHPP's from the same HPP, as described above. The points of the smaller one will be a subset of the larger one. That w.p.1 order implies the corresponding weak stochastic order.

3. Coupling

Interestingly, it is possible to go the other way. Given stochastic order, it is always possible to do a special construction and get w.p.1 order for new random variables on a new sample space. See §9.2 and especially Proposition 9.2.2 of Ross. That is, if $X \le_{st} Y$, then there exist new random variables $\tilde{X}$ and $\tilde{Y}$ on a new probability space such that $\tilde{X}$ is distributed the same as X, $\tilde{Y}$ is distributed the same as Y, and

$$P(\tilde{X} \le \tilde{Y}) = 1. \tag{3}$$

This is like Skorohod's theorem in homework 1 (concerning convergence of random variables, both for the case of real-valued random variables, but the results extend).

Here is an application: From this representation, it is easy to see that (1) implies (2). We first carry out the w.p.1 construction to get (3), from which it follows that

$$P(h(\tilde{X}) \le h(\tilde{Y})) = 1, \tag{4}$$

which in turn immediately implies that

$$E[h(\tilde{X}) \le E[h(\tilde{Y})]$$

But, since $\tilde{X}$ is distributed the same as X, and $\tilde{Y}$ is distributed the same as Y, we also necessarily also have (2). The special construction for NHPP's and HPP's directly produces such a w.p.1 construction.

In general stochastic comparisons can be very useful, e.g., see Chapter 9 of Ross and my paper and references there.

4. Homework 5

We went over homework 5 in class. The solutions are posted. The main point is that the probelms tend to be quite easy if you see a good approach, exploiting the right property of the PP or NHPP.