IEOR 6711: Stochastic Models I

Fall 2013, Professor Whitt, Thursday, October 15

The Excess, the Age and the Lifetime

1 The Inspection Paradox

The inspection paradox is discussed on pages 116-118 in §3.4.1, §4.2 and Example 3.6(B) of the blue Ross book and in Section 7.7 of the Green Ross book.

We first consider Homework Exercise 7.51 from the Green Ross book:

7.51. In 1984 the country of Morocco in an attempt to determine the average amount of time that tourists spend in that country on a visit tried two different sampling procedures. In one, they questioned randomly chosen tourists as they were leaving the country; in the other, they questioned randomly chosen guests at hotels. The average visiting time of the 3017 tourists chosen from hotels was 17.8, whereas the average visiting time of the $12,063$ tourists questioned at departure was 9.0. Can you explain this discrepancy? Does it necessarily imply a mistake?

We assume that they only considered tourists who stayed in hotels when asking at airports. We assume that guests in hotels are asked for the full length of their visit, future plus past. Otherwise, the number for hotels should be even larger, if we had to correct for not counting the future.

We observed that these results are in fact entirely reasonable, because the sampling at departures is sampling at the departure epochs, whereas sampling at the hotels is sampling at an arbitrary continuous time in the middle of visits. This sampling problem is not directly a renewal process; it is actually a more complicated stochastic process, because the visits of different customers overlap in general. The number of tourists in the country can be thought of as a queueing process. It goes up one at each arrival epoch, and it goes down 1 at each departure epoch. We get a discrete-time view if we look at the lengths of stay of successive departures (and also if we look at successive arrivals), but we get a length-biased view if we look at the customers present at an arbitrary time. There are likely to be more long intervals crossing an arbitrary time.

2 The Residual Lifetime, the Age and the Lifetime

This phenomenon is easy to analyze in the relatively simple setting of a renewal process, where there is only a single point process. Important associated stochastic processes are the age (or backward excess) $A(t)$, residual lifetime (or forward excess) $Y(t)$ and total lifetime $L(t)$ at time t in a renewal process. These quantities are defined in terms of the renewal (counting) process $N \equiv \{N(t) : t \geq 0\}$ and associated renewal times $S_n \equiv X_1 + \cdots X_n$, $n \geq 1$ (with $S_0 \equiv 0$) by

$$\begin{aligned} A(t) &\equiv t - S_{N(t)}, \\ Y(t) &\equiv S_{N(t)+1} - t, \\ L(t) &\equiv Y(t) + A(t) = S_{N(t)+1} - S_{N(t)} = X_{N(t)+1}, \quad t \geq 0 \ . \end{aligned}$$

2.1 Stochastic Order for the Lifetime

The lifetime observed at time t tends to be longer than a random interval between points, X_n. There is length-biased sampling. Let $\leq_{st}$ denote stochastic order, as in §9.1 of Ross.

Proposition 1 (*stochastic order for the lifetime*) *In a renewal process,*

$$L(t) = X_{N(t)+1} \geq_{st} X_1,$$

i.e.,

$$P(L(t) > x) \geq P(X_1 > x) \equiv F^c(x) \quad \textit{for all} \quad x \geq 0.$$

Proof. Condition on $S_{N(t)}$ and then uncondition. Observe that

$$P(X_{N(t)+1} > x | S_{N(t)} = t - s) \geq F^c(x) \quad \text{for all} \quad s > 0.$$

The conclusion follows by unconditioning: First note that

$$P(X_{N(t)+1} > x | S_{N(t)} = t - s) = P(X > x | X > s),$$

for all t and s with $0 \leq s \leq t$ and all $x \geq 0$. Then note that $P(X > x | X > s) = 1$ for $s > x$ and

$$P(X > x | X > s) = \frac{P(X > x)}{P(X > s)} \geq P(X > x) \quad \text{for} \quad 0 \leq s \leq x.$$

■

2.2 Continuous-Time Markov Processes

The processes $A \equiv \{A(t) : t \geq 0\}$ and $Y \equiv \{Y(t) : t \geq 0\}$ are continuous-time continuous-state Markov processes. (The conditional distribution of a future event after time t, given the present state at time t and past states prior to time t coincides with the conditional distribution of that future event after time t given only the present state at time t.)

The relevant fact for Chapter 3 of Ross is that these processes can all be analyzed by exploiting renewal theory.

3 Direct Application of the Key Renewal Theorem

To start with, we look at the limits of the probabilities $P(A(t) \leq x)$, $P(Y(t) \leq x)$ and $P(L(t) \leq x)$ as $t \to \infty$. We start by observing that the probabilities $P(A(t) \leq x)$, $P(Y(t) \leq x)$ and $P(L(t) \leq x)$ as functions of time t all satisfy renewal equations. This class was primarily a review of the two previous classes, doing new examples. For the following, see Sections 5-7 of the notes for for the last class.

The general renewal equation is

$$g(t) = h(t) + \int_0^t g(t-s)\, dF(s), \quad t \geq 0\ . \tag{1}$$

3.1 The Residual Life or (Forward) Excess

For example, we can let $g(t) \equiv P(Y(t) \le x)$. The first step in obtaining (1) in this context (if we were to prove the result (1) directly instead of just applying it) is to write

$$P(Y(t) \le x) = P(Y(t) \le x, X_1 > t) + P(Y(t) \le x, X_1 \le t), \tag{2}$$

where $P(A,B) \equiv P(A \cap B)$, i.e., the comma in (2) above means "and." Equation (2) is an elementary application of the law of total probabilities: The events are related by

$$\{Y(t) \le x\} = \{Y(t) \le x, X_1 > t\} \bigcup \{Y(t) \le x, X_1 \le t\}, \tag{3}$$

where

$$\{Y(t) \le x, X_1 > t\} \bigcap \{Y(t) \le x, X_1 \le t\} = \emptyset, \tag{4}$$

from which (2) follows trivially. In this context, we can apply the renewal equation in (1), writing

$$g(t) \equiv P(Y(t) \le x) \quad \text{and} \quad h(t) \equiv P(Y(t) \le x, X_1 > t) = F(t+x) - F(t).$$

We use the renewal property to get the second integral term in equation (1)

$$P(Y(t) \le x, X_1 \le t) = \int_0^t g(t-s)\, dF(s).$$

We then use the solution of the renewal equation

$$g(t) = h(t) + \int_0^t h(t-s)\, dm(s), \quad t \ge 0 \ , \tag{5}$$

where $m(t)$ is the renewal function. Equation (5) gives a characterization of $g(t) \equiv P(Y(t) \le x)$. Given the Laplace transform $\hat{f}(s) \equiv E[e^{-sX_1}]$, we can calculate the Laplace transform $\hat{g}(s)$ of $g(t)$. With numerical inversion we can then calculate $P(Y(t) \le x)$ for any t and x.

However, we obtain a nice simple story if we let t converge to ∞. We obtain a useful simple formula. In particular, we can use the key renewal theorem to deduce that, in general,

$$\lim_{t\to\infty} g(t) = \frac{\int_0^\infty h(s)\, ds}{E[X]} \quad \text{as} \quad t \to \infty \ , \tag{6}$$

provided that the function h is directly-Riemann integrable(DRI), which holds here. As a consequence, we get

$$\lim_{t\to\infty} P(Y(t) \le x) = \frac{\int_0^\infty (F(t+x) - F(t))\, dt}{E[X]} = \frac{\int_0^x (1 - F(t))\, dt}{E[X]} = F_e(x) \quad \text{as} \quad t \to \infty \ , \tag{7}$$

where F_e has the stationary-excess distribution.

3.2 Mean Values

Suppose we want to evaluate $E[Y(t)]$ instead of $P(Y(t) \le x)$. Essentially the same argument applies. Instead of equation (2), we write

$$E[Y(t)] = E[Y(t)1_{\{X_1 > t\}}] + E[Y(t)1_{\{X_1 \le t\}}] \tag{8}$$

and then we rewrite it as

$$g(t) \equiv E[Y(t)] = E[Y(t)|X_1 > t]P(X_1 > t) + E[Y(t)|X_1 \le t]P(X_1 \le t). \tag{9}$$

We then rewrite the second term as

$$E[Y(t)|X_1 \le t]P(X_1 \le t) = \int_0^t E[Y(t)|X_1 = s]dF(s) = \int_0^t E[Y(t-s)]dF(s) = \int_0^t g(t-s)dF(s)$$

and we rewrite the first term as

$$h(t) = E[Y(t)|X_1 > t]P(X_1 > t) = E[X_1 - t|X_1 > t]P(X_1 > t)$$

where $(x)^+ \equiv \max\{x, 0\}$. Then we apply the solution of the renewal equation and the key renewal theorem to get

$$g(t) \to \frac{\int_0^\infty h(t)\,dt}{E[X]} = \frac{E[X^2]}{2E[X]};$$

for the derivation in the last step, see the display above Proposition 3.4.6 on p. 120.

3.3 The Age

Suppose that we want to look at $P(A(t) \le x)$. Then we let $g(t) \equiv P(A(t) \le x)$. We proceed just as above. We get

$$g(t) \equiv P(A(t) \le x) = P(A(t) \le x, X_1 > t) + P(A(t) \le x, X_1 \le t).$$

we then apply (1) by writing

$$h(t) = P(A(t) \le x, X_1 > t) = P(X_1 > t)1_{\{t \le x\}} = F^c(t)1_{\{t \le x\}}$$

where $F^c(x) \equiv 1 - F(x)$ and 1_A is the indicator function of the event A, i.e., $1_A = 1$ on A (when A is true) and is 0 otherwise. Just as in the previous example, we identify the renewal equation, we apply its solution (which always exists) and then we apply the key renewal theorem, just as in (7), to get

$$\lim_{t \to \infty} P(A(t) \le x) = \frac{\int_0^\infty (F^c(t)1_{\{t \le x\}})\,dt}{E[X]} = \frac{\int_0^x F^c(t)\,dt}{E[X]} = F_e(x) \quad \text{as} \quad t \to \infty\ ,$$

where F_e again has the stationary-excess distribution.

3.4 The Lifetime

Suppose that we want to look at $P(L(t) > x)$. We need to consider

$$h(t) \equiv P(L(t) > x, X_1 > t) = P(X_1 > x \vee t),$$

where $x \vee t \equiv \max\{x, t\}$. Then

$$\begin{aligned} \int_0^\infty h(t)\,dt &= \int_0^\infty P(X_1 > x \vee t)\,dt = xP(X_1 > x) + \int_x^\infty P(X_1 > t)\,dt \\ &= \int_x^\infty t f(t)\,dt \end{aligned}$$

after doing an integration by parts on the second term

$$\int_a^b u dv = u(b)v(b) - u(a)v(a) - \int_a^b v du,$$

with $u \equiv F^c(t)$ and $v \equiv 1$.

4 Special Cases of Alternating Renewal Process

In Section 3.4.1 Ross observes that these results for the excess and the age can in fact be viewed as special cases of the corresponding result for alternating renewal processes, which we did in the last class. That is Theorem 3.4.4 in the book. For that application we divide each cycle of length X_i into two parts. The first part is the portion on which $Y(t) \leq x$, while the second part is the portion where $Y(t) > x$. These two intervals within each cycle are dependent, but the sequence of ordered pairs associated with successive cycles is a sequence of i.i.d. random vectors. The lifetime can be treated in the same way; see p. 117 of Ross.

5 Long-Run Averages: Application of the Renewal Reward Theorem

We can also apply the renewal reward theorem to establish corresponding results for the long run proportion of time that the excess is less than x and so forth. See Section 3.6 of the book. Instead of focusing on the limiting probability

$$\lim_{t\to\infty} P(Y(t) \leq x) = F_e(x),$$

established above, we are concerned with the weaker result

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_{\{Y(s)\leq x\}}\, ds = F_e(x). \tag{10}$$

The limit of the time average can be identified as the average reward per cycle divided by the average length of a cycle. See Theorem 3.6.1 and the notes of 10/7.

The idea is to introduce rewards associated with each cycle. The reward associated with cycle i is

$$R_i \equiv \int_{S_{i-1}}^{S_i} 1_{\{Y(t)\leq x\}}\, dt.$$

Note that R_i is distributed the same as R_1, where $a \wedge b \equiv \min\{a, b\}$,

$$R_1 = (X_1 \wedge x) \quad \text{and} \quad E[R_1] = E[(X_1 \wedge x)] = \int_0^\infty P((X_1 \wedge x) > t)\, dt = \int_0^x P((X_1 > t)\, dt = F_e(x).$$

Alternatively, we could focus on

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_{\{Y(s)> x\}}\, ds = 1 - F_e(x). \tag{11}$$

The reward associated with cycle i is

$$R_i \equiv \int_{S_{i-1}}^{S_i} 1_{\{Y(t)> x\}}\, dt.$$

Note that R_i is distributed the same as R_1, where $(x)^+ \equiv \max\{x, 0\}$,

$$R_1 = (X_1 - x)^+ \quad \text{and} \quad E[R_1] = E[(X_1 - x)^+] = \int_0^\infty P((X_1 - x)^+ > t)\, dt = \int_x^\infty P(X_1 > t)\, dt = 1 - F_e(x).$$

Actually there is one missing detail. The time averages in (10) and (11) involve an extra remainder term. That is, we can write

$$|\frac{1}{t}\int_0^t 1_{\{Y(s)\le x\}}\, ds - \frac{1}{t}\sum_{i=1}^{N(t)} R_i| \le \frac{R_{N(t)+1}}{t},$$

where R_i is an appropriate reward for the i^{th} cycle, while $R_{N(t)+1}$ is the reward associated with the cycle in progress at time t. This extra term $R_{N(t)+1}/t$ at the end can be shown to be asymptotically negligible. See Theorem 3.6.1 and p. 135 of the book.