

IEOR 6711: Stochastic Models I
First Midterm Exam, Chapters 1-2, October 7, 2008

Justify your answers; show your work.

1. A sequence of Events. (10 points)

Let $\{B_n : n \geq 1\}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$.

- (a) Explain what that means.
- (b) Under what useful condition(s) does the sequence of real numbers $\{P(B_n) : n \geq 1\}$ converge to a limit?
- (c) What is the limit in part (b) above when the conditions are satisfied?
- (d) Prove the result stated in parts (b) and (c) above.

2. Random Variables and MGF's. (10 points)

Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, where Ω is a finite set.

- (a) Define the probability distribution of X
- (b) Define the moment generating function (mgf) of X .
- (c) Give three representations for $E[h(X)]$, where h is a real-valued function of a real variable.
- (d) Give an expression for $E[X^3]$ in terms of the mgf of X .

3. A Sequence of Random Variables. (10 points)

Let $\{X_n : n \geq 1\}$ be a sequence of random variables such that

$$P(X_n = n^4) = \frac{1}{n^2} = 1 - P(X_n = 17 + (-1^n/\sqrt{n})), \quad n \geq 1.$$

- (a) Determine whether or not the sequence $\{X_n : n \geq 1\}$ converges for each of the following four modes of convergence:
 - (i) convergence in distribution
 - (ii) convergence in the mean
 - (iii) convergence in probability
 - (iv) convergence with probability 1 (w.p.1)
- (b) For those cases in which the sequence above converges, identify the limit.

4. Two Exponential Random Variables (25 points)

Let X_1 and X_2 be two independent exponentially distributed random variables, with means $E[X_1] = 1$ and $E[X_2] = 1/2$. Let

$$Y \equiv \text{minimum}\{X_1, X_2\} \quad \text{and} \quad Z \equiv \text{maximum}\{X_1, X_2\}.$$

Find the following quantities:

- (a) the mean of Y : $E[Y]$,
- (b) the mean of Z : $E[Z]$,
- (c) the cdf of Y : $F_Y(t) \equiv P(Y \leq t)$, $t \geq 0$,
- (d) the cdf of Z : $F_Z(t) \equiv P(Z \leq t)$, $t \geq 0$,
- (e) $E[e^{Z/2}]$,
- (f) the covariance of Y and Z : $\text{cov}(Y, Z)$,
- (g) the conditional probability $P(Z > 6|Y = 1)$.

5. Mind Over Matter (25 points)

Olivia claims that she has supernatural powers. She claims that she has the ability, by the power of her mental concentration, coupled with divine guidance, to increase the chance that three coins tossed together come out the same, either all three heads or all three tails. Suppose that we conduct a series of experiments to test Olivia's talent. In each experiment we toss three coins, and see whether or not the three outcomes are identical (i.e., if the outcome is either HHH or TTT).

- (a) Suppose that we repeat the experiment 4,800 times. Suppose that Olivia is successful (the outcome is either HHH or TTT) 1321 times out of 4,800. Does that result present strong evidence that Olivia actually does not possess this special talent? Does that result present strong evidence that she actually does possess this special talent? Why or why not?
- (b) State a theorem supporting your analysis in part (a).
- (c) Prove the theorem stated in part (b).

6. Rembrandt's Long-Lost Painting (20 points)

There is great excitement in the art world, because Rembrandt's long-lost painting of his Aunt Mabel has been miraculously discovered in the attic of an apartment of a Columbia engineering professor. This painting has been placed on view in the Museum of Modern Art. You get to help plan for the exhibit. You want to ensure that there is enough space to accommodate the many enthusiastic visitors.

Suppose that, on each day, visitors will come to see the painting according to a nonhomogeneous Poisson process with increasing arrival rate $\lambda(t) \equiv 50t$ per hour, $0 \leq t \leq 4$, during the allotted four-hour morning viewing period. Suppose that a visitor coming to the painting at time t will decide to stop and study the painting with an increasing probability $p(t) \equiv t/5$, $0 \leq t \leq 4$, independently of what all the other visitors do. (Otherwise the visitor will continue walking right on by, and look at other exhibits.) Moreover, each visitor who does decide to stop

does so for a random length of time, which is uniformly distributed between 0 and $1/2$ hour, i.e., is uniformly distributed over the interval $[0, 1/2]$. These random durations are mutually independent for the different visitors. Based on these (highly dubious) assumptions, answer the following questions:

(a) What is the distribution of the total number of visitors to see the painting in the first two hours (including both those visitors who stop and those who do not)?

(b) Let $V(t)$ be the number of visitors viewing the painting at time t . (That includes those who have come before time t and stopped, but have not left already.) What is the probability distribution of $V(3)$?

(c) What is the covariance between $V(2)$ and $V(4)$?