IEOR 6711: Stochastic Models I

SOLUTIONS to the First Midterm Exam, October 7, 2008

Justify your answers; show your work.

1. A sequence of Events. (10 **points)**

Let $\{B_n : n \geq 1\}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$.

(a) Explain what that means; i.e., what is an "event"?

See pages 1-3 of the book. No need for a lot of words. There are two points:

$$B_n \subseteq \Omega \quad \text{and} \quad B_n \in \mathcal{F}.$$

That is, B_n is a subset of the sample space, which is also measurable. In notation, it is important to distinguish clearly between $\in$ and $\subseteq$.

(b) Under what useful condition(s) does the sequence of real numbers $\{P(B_n) : n \geq 1\}$ converge to a limit?

Convergence holds under the conditions: the sequence is increasing or decreasing. That needs to be defined. Increasing means that $B_n \subseteq B_{n+1}$ for all n. Decreasing is $B_n \supseteq B_{n+1}$ for all n. See page 2.

(c) What is the limit in part (b) above when the conditions are satisfied?

The infinite union or intersection is the limit of the sets when the sets are increasing or decreasing. Then we have

$$lim_{n\to\infty} P(B_n) = P\left(lim_{n\to\infty} B_n\right),$$

where, for the increasing case,

$$lim_{n\to\infty} B_n \equiv \bigcup_{n=1}^{\infty} B_n.$$

(d) Prove the result stated in parts (b) and (c) above.

See pages 2-3 of the book. I am asking for the proof of Proposition 1.1.1 on page 2. The idea is to exploit the axiom of countable additivity. What we are establishing is that "continuity is implied by countable additivity." They are actually equivalent, given the other axioms.

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2. Random Variables and MGF's. (10 points)

Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, where Ω is a finite set.

(a) Define the probability distribution of X

See the lecture notes of September 4. See §2 of those notes. The probability distribution of X is a probability measure on $\mathbb{R}$, denoted by P_X say, defined in terms of the underlying probability measure P by

$$P_X(A) \equiv P(X^{-1}(A)) \equiv P(\{\omega \in \Omega : X(\omega) \in A\}) \quad \text{for} \quad A \subseteq \mathbb{R}.$$

(b) Define the moment generating function (mgf) of X.

See §1.4 on page 15 of the book. See the notes for September 11. The mgf is

$$\psi_X(t) \equiv E[e^{tX}].$$

For each random variable X, we get a function of t.

(c) Give three representations for $E[h(X)]$, where h is a real-valued function of a real variable.

See page 5 of the lecture notes for September 4. This should be clean. The idea is given by the figures there.

(d) Give an expression for $E[X^3]$ in terms of the mgf of X.

See top of page 16 of the book. It is the third derivative of the mgf defined in (b) evaluated at $t = 0$.

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3. A Sequence of Random Variables. (10 points)

Let $\{X_n : n \geq 1\}$ be a sequence of random variables such that

$$P(X_n = n^4) = \frac{1}{n^2} = 1 - P(X_n = 17 + (-1^n/\sqrt{n})), \quad n \geq 1.$$

(a) Determine whether or not the sequence $\{X_n : n \geq 1\}$ converges for each of the following four modes of convergence:

(i) convergence in distribution

(ii) convergence in the mean

(iii) convergence in probability

(iv) convergence with probability 1 (w.p.1)

(b) For those cases in which the sequence above converges, identify the limit.

The sequence fails to converge in the mean. It is easy to see that $E[X_n] \to \infty$ as $n \to \infty$. Ignoring the second complicated part of X_n, we have $n^4 \times (1/n^2) = n^2 \to \infty$. On the other hand, we do have converges w.p.1, as can be demonstrated by applying the Borel-Centelli lemma. This needs to be done carefully. Let Y_n be the second complicated deterministic piece; i.e., $Y_n = 17 + (-1)^n/\sqrt{n}$, $n \geq 1$. (Y_n is deterministic.) Then

$$P(|X_n - Y_n| > 0 \quad \text{infinitely often}) = 0,$$

by Borel-Cantelli, because

$$\sum_{n=1}^{\infty} P(|X_n - Y_n| > 0) = \sum_{n=1}^{\infty} (1/n^2) < \infty.$$

Trivially, $Y_n \to 17$ as $n \to \infty$ w.p.1. (because it is just a deterministic sequence). Hence

$$X_n = Y_n + (X_n - Y_n) \to 17 + 0 = 17 \quad \text{as} \quad n \to \infty \quad \text{w.p.1.}$$

It is good to do this carefully and cleanly. That convergence w.p.1 we have established implies convergence in probability, which in turn implies convergence in distribution. The limit is a random variable X with $P(X = 17) = 1$. See the lecture notes for September 9.

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4. Two Exponential Random Variables (25 points)

Let X_1 and X_2 be two independent exponentially distributed random variables, with means $E[X_1] = 1$ and $E[X_2] = 1/2$. Let

$$Y \equiv \text{minimum}\{X_1, X_2\} \quad \text{and} \quad Z \equiv \text{maximum}\{X_1, X_2\}.$$

Find the following quantities:

(a) the mean of Y: $E[Y]$,

From (c) below, we know that Y is exponential with a rate equal to the sum of the rates:

$$\lambda = \lambda_1 + \lambda_2 = 1 + 2 = 3.$$

Thus $E[Y] = 1/3$.

(b) the mean of Z: $E[Z]$,

Notice that $Y+Z = X_1+X_2$. Hence $E[Z] = E[X_1]+E[X_2]-E[Y] = 1+(1/2)-(1/3) = 7/6$.

(c) the cdf of Y: $F_Y(t) \equiv P(Y \le t), \quad t \ge 0$,

$$P(Y > t) = P(X_1 > t, X_2 > t) = P(X_1 > t)P(X_2 > t) = e^{-3t}.$$

(d) the cdf of Z: $F_Z(t) \equiv P(Z \le t), \quad t \ge 0$,

Paralleling part (c),

$$P(Z \le t) = P(X_1 \le t, X_2 \le t) = P(X_1 \le t)P(X_2 \le t) = (1 - e^{-t})(1 - e^{-2t}).$$

We can do more algebra to get a clean expression: $1 - e^{-t} - e^{-2t} + e^{-3t}$.

(e) $E\left[e^{Z/2}\right]$,

$$E\left[e^{Z/2}\right] = \int_0^\infty e^{x/2} f_Z(x)\, dx.$$

Easy to get pdf of Z from part (d). Final answer 32/15.

(f) the covariance of Y and Z: $cov(Y, Z)$,

Many ways to do this. One way is to write, $Z \stackrel{\text{d}}{=} Y + W$, where W is independent of Y. In particular, W is hyperexponential (H_2), i.e., a mixture of two exponential random variables, being an exponential with mean 1/2 with probability 1/3 and an exponential with mean 1 with probability 2/3. Hence, using fundamental relations, we get

$$Cov(Y, Z) = Cov(Y, Y+W) = Cov(Y, Y)+Cov(Y, W) = Var(Y)+0 = Var(Y) = (1/3)^2 = 1/9.$$

(g) the conditional probability $P(Z > 6|Y = 1)$.

Given our analysis in part (f) above,

$$P(Z > 6|Y = 1) = P(W > 5) = (2/3)e^{-5} + (1/3)e^{-10}.$$

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5. Mind Over Matter (25 points)

Olivia claims that she has supernatural powers. She claims that she has the ability, by the power of her mental concentration, coupled with divine guidance, to increase the chance that three coins tossed together come out the same, either all three heads or all three tails. Suppose that we conduct a series of experiments to test Olivia's talent. In each experiment we toss three coins, and see whether or not the three outcomes are identical (i.e., if the outcome is either HHH or TTT).

(a) Suppose that we repeat the experiment $4,800$ times. Suppose that Olivia is successful (the outcome is either HHH or TTT) 1321 times out of $4,800$. Does that result present strong evidence that Olivia actually does not possess this special talent? Does that result present strong evidence that she actually does possess this special talent? Why or why not?

Consider the null hypothesis that Olivia does not have special talent. Under that hypothesis, we regard successive experiments as Bernoulli trials with probability $p = 1/4$ of success. The expected number of successes in $n = 4800$ trials is thus 1200. The total number of successes would then have a binomial distribution with $n = 4800$ and $p = 1/4$. The idea then is to use a normal approximation. We see that the variance is $np(1-p) = 900$, which has an easy-to-calculate square root of 30, the standard deviation. We see that the test result of 1321 is slightly more than 4 standard deviations above the mean 1200. We deduce that we cannot reject the null hypothesis, under most criteria (95% or 99%). The experiment does support the conclusion that Olivia does have special powers, which we might have anticipated from non-mathematical considerations.

(b) State a theorem supporting your analysis in part (a).

This is the central limit theorem (CLT). Remember that the conditions include finite first two moments as well as sum of i.i.d. random variables.

(c) Prove the theorem stated in part (b).

See lecture notes of September 11, toward end. The proof is an application of characteristic functions.

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6. Rembrandt's Long-Lost Painting (20 points)

There is great excitement in the art world, because Rembrandt's long-lost painting of his Aunt Mabel has been miraculously discovered in the attic of an apartment of a Columbia engineering professor. This painting has been placed on view in the Museum of Modern Art. You get to help plan for the exhibit. You want to ensure that there is enough space to accommodate the many enthusiastic visitors.

Suppose that, on each day, visitors will come to see the painting according to a nonhomogeneous Poisson process with increasing arrival rate $\lambda(t) \equiv 50t$ per hour, $0 \leq t \leq 4$, during the allotted four-hour morning viewing period. Suppose that a visitor coming to the painting at time t will decide to stop and study the painting with an increasing probability $p(t) \equiv t/5$, $0 \leq t \leq 4$, independently of what all the other visitors do. (Otherwise the visitor will continue walking right on by, and look at other exhibits.) Moreover, each visitor who does decide to stop does so for a random length of time, which is uniformly distributed between 0 and 1/2 hour, i.e., is uniformly distributed over the interval $[0, 1/2]$. These random durations are mutually independent for the different visitors. Based on these (highly dubious) assumptions, answer the following questions:

(a) What is the distribution of the total number of visitors to see the painting in the first two hours (including both those visitors who stop and those who do not)?

The general idea for this problem is that we are in the setting of a nonhomogeneous Poisson process and the $M_t/GI/\infty$ queueing model, as in the 1993 physics paper discussed on September 23. The number of arrivals in the interval $[0, 2]$ is Poisson with mean

$$m(2) \equiv \int_0^2 \lambda(t)\, dt = \int_0^2 50t\, dt = 100.$$

(b) Let $V(t)$ be the number of visitors viewing the painting at time t. (That includes those who have come before time t and stopped, but have not left already.) What is the probability distribution of $V(3)$?

Here we want the formula for the number of customers in the system in the $M_t/GI/\infty$ queueing model. By Theorem 1 ofg the physics paper, this number has a Poisson distribution with mean

$$m(3) = \int_{2.5}^{3} (50s)(s/5)(1 - 2(3-s))\, ds = \int_{2.5}^{3} 10s^2(1 - 2(3-s))\, ds = \frac{965}{48}.$$

(c) What is the covariance between $V(2)$ and $V(4)$?

Clearly $V(2)$ and $V(4)$ are independent, because all customers in the system at time 2 are necessarily gone by time 4, because the length of stay is uniform in the interval $[0, 1/2]$. Hence the covariance is 0.
