IEOR 6711: Stochastic Models I

First Midterm Exam, Chapters 1-2, October 10, 2010

Justify your answers; show your work.

1. Exponential Random Variables (23 points)

Let X_1 and X_2 be independent exponential random variables with means $E[X_1] \equiv 1/\lambda_1$ and $E[X_2] \equiv 1/\lambda_2$. Let

$$M_1 \equiv \min\{X_1, X_2\} \quad \text{and} \quad M_2 \equiv \max\{X_1, X_2\}.$$

Compute and derive the following quantities:

(a) $P(M_1 > t, M_1 = X_1)$,

(b) $P(M_1 > t | M_1 = X_1)$,

(c) $Var(M_1 + M_2)$,

(d) $P(M_1 > t_1, M_2 > t_2)$.

(e) Now suppose that $\lambda_1 = \lambda_2 = \lambda$. Compute the *probability density function* (pdf) of $X_1 - X_2$.

2. Characteristic Functions and Cauchy Random Variables. (24 points)

For a random variable Y with pdf $f_Y(x)$, let $\phi_Y(\theta)$ be its *characteristic function* (cf), defined by

$$\phi_Y(\theta) \equiv E\left[e^{i\theta Y}\right] = \int_{-\infty}^{+\infty} e^{i\theta x} f_Y(x)\, dx \ ,$$

where $i \equiv \sqrt{-1}$. We review two properties of cf's: (i) When the cf ϕ_Y is an integrable function, the pdf f_Y can be recovered from the cf by the *inversion formula*

$$f_Y(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \phi_Y(\theta)\, d\theta \ . \tag{1}$$

(ii) If $E[|Y|^k] < \infty$, then ϕ_Y has a continuous k^{th} derivative given by

$$\phi_Y^{(k)}(\theta) = \int_{-\infty}^{+\infty} (ix)^k e^{ix\theta} f_Y(x)\, dx.$$

(a) In the setting of Problem 1 (e), compute the cf of $X_1 - X_2$.

(b) Suppose that Y is a Cauchy random variable (centered at 0) with positive scale parameter σ; i.e., suppose that Y has pdf

$$f_Y(x) \equiv \frac{\sigma}{\pi(\sigma^2 + x^2)}, \quad -\infty < x < +\infty \ . \tag{2}$$

Show that Y has cf $\phi_Y(\theta) = e^{-\sigma|\theta|}$. (Hint: Use parts 1 (e) and 2 (a) with (1).)

(c) What does Property (ii) above and part (b) imply about $E[|Y|]$?

(d) Suppose that $Y_1, Y_2, \ldots$ are i.i.d. random variables with the Cauchy pdf in (2) and let $\bar{Y}_n \equiv (Y_1 + \cdots + Y_n)/n$ for $n \geq 1$. Use part (b) to describe the asymptotic behavior of $\bar{Y}_n$ and $\sqrt{n}\bar{Y}_n$ as $n \to \infty$.

3. More exponential random variables. (**28 points**)

Let $\{Y_n : n \geq 1\}$ be a sequence of i.i.d. (independent and identically distributed) exponential random variables, each having mean 1. For $n \geq 4$, let $X_n = 1$ if $Y_n = \max\{Y_n, Y_{n-1}, Y_{n-2}, Y_{n-3}\}$; otherwise, let $X_n = 0$. Also let $X_1 = X_2 = X_3 = 0$. For $n \geq 1$, let $Z_n = n^2$ if $Y_{n^2} = \max\{Y_1, Y_2, \ldots, Y_{n^2-1}, Y_{n^2}\}$.

(a) (2 points) Determine the mean and variance of X_n, and the covariance $cov(X_n, X_{n+1})$ for $n \geq 4$.

(b) (2 points) Determine the mean and variance of Z_n, and the covariance $cov(Z_n, Z_{n+1})$.

(c) (6 points) For $n \geq 1$, Let $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Prove or disprove: $\bar{X}_n$ converges w.p.1 (with probability one) to a finite limit c. If the limit exists, then identify the constant c.

(d) (6 points) In the setting of part (c), Prove or disprove: There exists a finite constant c such that $E(\bar{X}_n - c)^2 \to 0$ as $n \to \infty$. If the limit exists, then identify the constant c.

(e) (6 points) For $n \geq 1$, Let $\bar{Z}_n = (Z_1 + \cdots + Z_n)/n$. Prove or disprove: $\bar{Z}_n$ converge w.p.1 to a finite limit c. If the limit exists, then identify the constant c.

(f) (6 points) In the setting of part (e), Prove or disprove: There exists a finite constant c such that $E(\bar{Z}_n - c)^2 \to 0$ as $n \to \infty$. If the limit exists, then identify the constant c.

4. The Columbia Space Company (25 points)

Columbia University has decided to start the Columbia Space Company, which will launch satellites from its planned Manhattanville launch site beginning in 2013, referred to henceforth as time 0. Allowing for steady growth, the Columbia Space Company plans to launch satellites at an increasing rate, beginning at time $t = 0$. Specifically, they anticipate that they will launch satellites according to a nonhomogeneous Poisson process with rate $\lambda(t) = 2t$ satellites per year for $t \geq 0$. Suppose that the successive times satellites stay up in space are independent random variables, each exponentially distributed with mean 2 years.

(a) What is the probability (according to this model) that no satellites will actually be launched during the first three years (between times $t = 0$ and $t = 3$)?

(b) What is the probability that precisely 7 satellites will be launched during the second year (between times $t = 1$ and $t = 2$)?

(c) Let $S(t)$ be the number of satellites in space at time t. Give an expression for the probability distribution of $S(6)$?

(d) Let $R(t)$ be the number of satellites that have been launched and have returned to earth in $[0, t]$. Give an expression for the joint probability $P(S(6) = 7, R(6) = 8)$.

(e) Give an expression for the covariance $Cov(S(6), S(8))$.