IEOR 6711: Stochastic Models I

SOLUTIONS to First Midterm Exam, October 10, 2010

Justify your answers; show your work.

1. Exponential Random Variables (23 points)

Let X_1 and X_2 be independent exponential random variables with means $E[X_1] \equiv 1/\lambda_1$ and $E[X_2] \equiv 1/\lambda_2$. Let

$$M_1 \equiv \min\{X_1, X_2\} \quad \text{and} \quad M_2 \equiv \max\{X_1, X_2\}.$$

Compute and derive the following quantities:

(a) $P(M_1 > t, M_1 = X_1)$,

We start out establishing basic properties of exponential distribution, as on the concise summary page.

$$\begin{aligned} P(M_1 > t, M_1 = X_1) &= \int_t^\infty f_{X_1}(x) F_2^c(x)\, dx \\ &= \int_t^\infty \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 x}\, dx \\ &= \left(e^{-(\lambda_1+\lambda_2)t}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) \\ &= P(M_1 > t)P(M_1 = X_1), \end{aligned} \tag{1}$$

showing independence of the two marginal distributions in the last line.

(b) $P(M_1 > t | M_1 = X_1)$,

By part (a), $P(M_1 > t | M_1 = X_1) = P(M_1 > t) = e^{-(\lambda_1+\lambda_2)t}$.

(c) $Var(M_1 + M_2)$,

Since $M_1 + M_2 = X_1 + X_2$, $Var(M_1 + M_2) = \lambda_1^{-2} + \lambda_2^{-2}$.

(d) $P(M_1 > t_1, M_2 > t_2)$.

There are two cases. The main case is $t_1 < t_2$. If $t_1 \ge t_2$, then

$$P(M_1 > t_1, M_2 > t_2) = P(M_1 > t_1) = e^{-(\lambda_1+\lambda_2)t_1}.$$

Hence, assume that $t_1 < t_2$. It is convenient to rewrite, getting

$$P(M_1 > t_1, M_2 > t_2) = P(M_1 > t_1) - P(M_1 > t_1, M_2 \le t_2).$$

Then

$$\begin{aligned} P(M_1 > t_1, M_2 \le t_2) &= P(t_1 < X_1 < t_2)P(t_1 < X_2 < t_2) \\ &= (e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2})(e^{-\lambda_2 t_1} - e^{-\lambda_2 t_2}) \end{aligned}$$

so that

$$\begin{aligned} P(M_1 > t_1, M_2 > t_2) &= e^{-(\lambda_1+\lambda_2)t_1} - (e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2})(e^{-\lambda_2 t_1} - e^{-\lambda_2 t_2}) \\ &= e^{-\lambda_1 t_1 - \lambda_2 t_2} + e^{-\lambda_1 t_2 - \lambda_2 t_1} - e^{-\lambda_1 t_2 - \lambda_2 t_2}. \end{aligned}$$

(e) Now suppose that $\lambda_1 = \lambda_2 = \lambda$. Compute the *probability density function* (pdf) of $X_1 - X_2$.

This is a basic exercise is convolution. When $\lambda_1 = \lambda_2 = \lambda$ the pdf of $X_1 - X_2$ is the symmetric bilateral exponential distribution $f(x) = (\lambda/2)e^{-\lambda|x|}$ on the entire real line. In detail, for $x > 0$,

$$f_{X_1 - X_2}(x) = \int_x^\infty f_{X_1}(y) f_{X_2}(y - x)\, dy = \frac{\lambda}{2} e^{-\lambda x}.$$

The reasoning is essentially the same for $x < 0$.

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2. Characteristic Functions and Cauchy Random Variables. (24 points)

For a random variable Y with pdf $f_Y(x)$, let $\phi_Y(\theta)$ be its *characteristic function* (cf), defined by

$$\phi_Y(\theta) \equiv E\left[e^{i\theta Y}\right] = \int_{-\infty}^{+\infty} e^{i\theta x} f_Y(x)\, dx \ ,$$

where $i \equiv \sqrt{-1}$. We review two properties of cf's: (i) When the cf ϕ_Y is an integrable function, the pdf f_Y can be recovered from the cf by the *inversion formula*

$$f_Y(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \phi_Y(\theta)\, d\theta \ . \tag{2}$$

(ii) If $E[|Y|^k] < \infty$, then ϕ_Y has a continuous k^{th} derivative given by

$$\phi_Y^{(k)}(\theta) = \int_{-\infty}^{+\infty} (ix)^k e^{ix\theta} f_Y(x)\, dx.$$

(a) In the setting of Problem 1 (e), compute the cf of $X_1 - X_2$.

By elementary calculations,

$$\phi_{X_1}(\theta) = \frac{\lambda}{\lambda - i\theta} \quad \text{and} \quad \phi_{-X_2}(\theta) = \frac{\lambda}{\lambda + i\theta}.$$

Hence

$$\phi_{(X_1 - X_2)}(\theta) = \phi_{X_1}(\theta)\phi_{-X_2}(\theta) = \frac{\lambda^2}{\lambda^2 + \theta^2}.$$

(b) Suppose that Y is a Cauchy random variable (centered at 0) with positive scale parameter σ; i.e., suppose that Y has pdf

$$f_Y(x) \equiv \frac{\sigma}{\pi(\sigma^2 + x^2)}, \quad -\infty < x < +\infty \ . \tag{3}$$

Show that Y has cf $\phi_Y(\theta) = e^{-\sigma|\theta|}$. (Hint: Use parts 1 (e) and 2 (a) with (2).)

Given the answers to parts 1 (e) and 2 (a), we can directly apply the inversion formula, as suggested. This is a standard "duality" for characteristic functions. See Section XV.2 of Feller, vol. II. These are examples 7 and 8 in Table 1 on p. 503. By the *inversion formula*

$$\begin{aligned}
f_Y(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\theta x}\phi_Y(\theta)\, d\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\theta x} e^{-\sigma|\theta|}\, d\theta \\
&= \frac{1}{\sigma\pi}\int_{-\infty}^{+\infty} e^{-i\theta x}\frac{\sigma}{2} e^{-\sigma|\theta|}\, d\theta \\
&= \left(\frac{1}{\sigma\pi}\right)\left(\frac{\sigma^2}{\sigma^2 + (-x)^2}\right) \\
&= \frac{\sigma}{\pi(\sigma^2 + x^2)},
\end{aligned}$$

where the fourth line follows from part (a). Notice that the integral has the same form with $-x$ here playing the role of θ in part (a).

(c) What does Property (ii) above and part (b) imply about $E[|Y|]$?

This question applies basic calculus or real analysis. The absolute value appearing in the cf implies that the cf is not differentiable. Thus, by the theorem quoted previously, the mean of $|Y|$, $E[|Y|]$, must not be finite. It is not difficult to see that the mean must be infinite directly, because

$$x^2 f_Y(x) \to \sigma/\pi \quad \text{as} \quad x \to \pm\infty.$$

Hence, $xf_Y(x) = O(1/x)$, so that we must have

$$\int_0^{\infty} x f_Y(x)\, dx = \int_{-\infty}^{0} |x| f_Y(x)\, dx = \infty.$$

(d) Suppose that $Y_1, Y_2, \ldots$ are i.i.d. random variables with the Cauchy pdf in (3) and let $\bar{Y}_n \equiv (Y_1 + \cdots + Y_n)/n$ for $n \geq 1$. Use part (c) to describe the asymptotic behavior of $\bar{Y}_n$ and $\sqrt{n}\bar{Y}_n$ as $n \to \infty$.

From part (c), we know that the standard LLN does *not* apply: The SLLN requires that the random variables being added have finite mean. Hence, we do not have $\bar{Y}_n \Rightarrow E[Y] = 0$

as $n \to \infty$. (In particular, the mean of Y does not exist and so is not actually 0, despite the symmetry. Hence, $\bar{Y}_n$ does not converge to any constant as $n \to \infty$.) In fact, we can say much more: Using the cf, we immediately see that the average $\bar{Y}_n$ has a distribution independent of n, i.e., is the same as Y_1:

$$\phi_{\bar{Y}_n}(\theta) = \phi_Y(\theta/n)^n = (e^{-\sigma|\theta/n|})^n = e^{-\sigma|\theta|} = \phi_Y(\theta).$$

Hence, trivially, the average $\bar{Y}_n$ converges in distribution to the original Cauchy law, i.e.,

$$\bar{Y}_n \Rightarrow Y_1 \quad \text{as} \quad n \to \infty.$$

As a consequence, the limit exists (in distribution), but the limit has a nondegenerate distribution. Since, $\bar{Y}_n \Rightarrow Y_1$, it follows that $\sqrt{n}\bar{Y}_n$ diverges. In particular, we do *not* have the conventional central limit theorem holding here.

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3. More exponential random variables. (28 **points**)

Let $\{Y_n : n \geq 1\}$ be a sequence of i.i.d. (independent and identically distributed) exponential random variables, each having mean 1. For $n \geq 4$, let $X_n = 1$ if $Y_n = \max\{Y_n, Y_{n-1}, Y_{n-2}, Y_{n-3}\}$; otherwise, let $X_n = 0$. Also let $X_1 = X_2 = X_3 = 0$. For $n \geq 1$, let $Z_n = n^2$ if $Y_{n^2} = \max\{Y_1, Y_2, \ldots, Y_{n^2-1}, Y_{n^2}\}$.

(a) (2 points) Determine the mean and variance of X_n, and the covariance $cov(X_n, X_{n+1})$ for $n \geq 4$.

Since the distribution is continuous, the probability of any multiple values (ties) is zero. The ordering of the vector $(Y_1, \ldots, Y_n)$ can be represented as a random permutation of the vector $(1, 2, \ldots, n)$, where k denotes the k^{th} smallest. For example, for $n = 3$, if $Y_2 > Y_3 > Y_1$, then the outcome is $(1, 3, 2)$. The key property is that all $n!$ permutations are equally likely. There are $(n-1)!$ permutations with the maximum element in the right place. Each one is equally likely to be the maximum. Hence, $E[X_n] = E[X_n^2] = 1/4$, so that $Var(X_n) = 1/4 - (1/4)^2 = 3/16$.

The random variables X_n and X_{n+1} are dependent. We see that we have $X_n X_{n+1} = 1$ if and only if among 5 consecutive Y_i $(i = n-3, \ldots n+1)$, Y_{n+1} is the maximum, and then Y_n is the second largest. The first event happens with probability $1/5$. The second event, conditional on the first, happens with probability $1/4$. Hence

$$E[X_n X_{n+1}] = (1/5) \times (1/4) = 1/20.$$

so that

$$cov(X_n, X_{n+1}) = E[X_n X_{n+1}] - E[X_n]E[X_{n+1}] = \left(\frac{1}{20}\right) - \left(\frac{1}{4}\right)^2 = -1/80.$$

The successive X_n are slightly negatively correlated (consistent with intuition).

(b) (2 points) Determine the mean and variance of Z_n, and the covariance $cov(Z_n, Z_{n+1})$.

$$E[Z_n] = n^2(1/n^2) = 1 \quad \text{and} \quad E[Z_n^2] = n^4(1/n^2) = n^2,$$

so that

$$Var(Z_n) = E[Z_n^2] - (E[Z_n])^2 = n^2 - 1.$$

Given $Z_{n+1} = 1$, we know that $Y_{(n+1)^2}$ is the largest of the first $(n+1)^2$ values of Y_j. However, the remaining $((n+1)^2 - 1)!$ permutations of the remaining integers are equally likely. Hence,

$$P(Z_n = 1 | Z_{n+1} = 1) = P(Z_n = 1),$$

so that Z_n and Z_{n+1} are independent. Finally, independence implies that the variables are uncorrelated. Hence, $cov(Z_n, Z_{n+1}) = 0$.

(c) (6 points) For $n \geq 1$, Let $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Prove or disprove: $\bar{X}_n$ converges w.p.1 (with probability one) to a finite limit c. If the limit exists, then identify the constant c.

This is a minor variant of Problem 1.37 in the first homework. We exploit the fact that the sequence of random variables $\{X_{4k} : k \geq 1\} \equiv X_4, X_8, X_{12}, \ldots$ are i.i.d. with $P(X_{4k} = 1) = 1/4! = 1/24 = 1 - P(X_{4k} = 0)$. Similarly, the sequences $\{X_{4k+1}; k \geq 1\} \equiv X_5, X_9, \ldots$, $\{X_{4k+2} : k \geq 1\} \equiv X_6, X_{10}, X_{14}, \ldots$ and $\{X_{4k+3} : k \geq 1\} \equiv X_7, X_{11}, X_{15}, \ldots$ are each i.i.d. with the same distribution. Hence, the strong law of large numbers (SLLN) applies to each of these subsequences separately. That in turn implies that the SLLN holds for the entire sequence, with $c = 1/4$. More generally, this example is a special case of m-dependence, a form of weak dependence.

(d) (6 points) In the setting of part (c), Prove or disprove: There exists a finite constant c such that $E(\bar{X}_n - c)^2 \to 0$ as $n \to \infty$. If the limit exists, then identify the constant c.

The limit does exist with $c = 1/4$, which is $E[X_n]$. In this case,

$$\begin{aligned}
E(\bar{X}_n - c)^2 &= Var(\bar{X}_n) = \frac{1}{n^2}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} Cov(X_j, X_k)\right) \\
&= \frac{1}{n^2}\left(O(1) + \sum_{j=1}^{n}(Var(X_j) + 6Cov(X_j, X_{j+1}))\right) \\
&= \frac{1}{n^2}\left(n\left(\frac{3}{16}\right) + 6n\left(\frac{-1}{80}\right) + O(1)\right) \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where $O(1)$ is the negligible error caused by the end effects.

(e) (6 points) For $n \geq 1$, let $\bar{Z}_n = (Z_1 + \cdots + Z_n)/n$. Prove or disprove: $\bar{Z}_n$ converge w.p.1 to a finite limit c. If the limit exists, then identify the constant c.

Here we apply the Borel-Cantelli lemma to show that $P(Z_n \neq 0 \quad \text{i.o.}) = 0$. That follows because
$$\sum_{n=1}^{\infty} P(Z_n \neq 0) = \sum_{n=1}^{\infty} P(Z_n = n^2) = \sum_{n=1}^{\infty} (1/n^2) < \infty.$$
Whenever, the events $\{Z_n \neq 0\}$ occur only finitely often, $\bar{Z}_n \to 0$. Hence, we have $\bar{Z}_n \to 0$ w.p.1.

(f) (6 points) In the setting of part (e), Prove or disprove: There exists a finite constant c such that $E(\bar{Z}_n - c)^2 \to 0$ as $n \to \infty$. If the limit exists, then identify the constant c.

There does not exist a constant c such that there is convergence in the mean-squared (in L_2). Note that
$$E(\bar{Z}_n - c)^2 = Var(\bar{Z}_n) + (c - E[Z_n])^2,$$
so that suffices to consider $c = E[\bar{Z}_n] = 1$ and it suffices to focus on $Var(\bar{Z}_n)$. However,
$$\begin{aligned} Var(\bar{Z}_n) &= \frac{1}{n^2} \sum_{k=1}^{n} Var(Z_k) = \frac{1}{n^2} \sum_{k=1}^{n} (k^2 - 1) \\ &= \frac{1}{n^2} \left(\frac{n(n+1)(2n+1)}{6} - n) \right) \\ &\to \infty \quad \text{as} \quad n \to \infty, \end{aligned}$$
being of order $O(n)$. Indeed, $Var(\bar{Z}_n)/n \to 1/3$ as $n \to \infty$ by the reasoning above.

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4. The Columbia Space Company (25 points)

Columbia University has decided to start the Columbia Space Company, which will launch satellites from its planned Manhattanville launch site beginning in 2013, referred to henceforth as time 0. Allowing for steady growth, the Columbia Space Company plans to launch satellites at an increasing rate, beginning at time $t = 0$. Specifically, they anticipate that they will launch satellites according to a nonhomogeneous Poisson process with rate $\lambda(t) = 2t$ satellites per year for $t \geq 0$. Suppose that the successive times satellites stay up in space are independent random variables, each exponentially distributed with mean 2 years.

(a) What is the probability (according to this model) that no satellites will actually be launched during the first three years (between times $t = 0$ and $t = 3$)?

Let $N(t)$ count the number of satellite launches in the interval $[0, t]$. The stochastic process $N \equiv \{N(t) : t \geq 0\}$ is directly a nonhomogeneous Poisson process, as in Section 2.4.
$$P(N(3) = 0) = \frac{e^{-m(3)} m(3)^0}{0!} = e^{-m(3)},$$
where
$$m(t) \equiv \int_0^t \lambda(u)\, du = \int_0^t 2u\, du = t^2. \tag{4}$$
Hence, $m(3) = 3^2 = 9$ and the answer is e^{-9}.

(b) What is the probability that precisely 7 satellites will be launched during the second year (between times $t = 1$ and $t = 2$)?

$$P(N(2) - N(1) = 7) = \frac{e^{-m(1,2)} m(1,2)^7}{7!},$$

where $m(s,t) \equiv m(t) - m(s)$ for

$$m(t) \equiv \int_0^t \lambda(u)\, du = \int_0^t 2u\, du = t^2. \tag{5}$$

Hence, $m(1,2) = 2^2 - 1^2 = 3$. The final answer is thus

$$P(N(2) - N(1) = 7) = \frac{e^{-3} 3^7}{7!}.$$

(b) Let $S(t)$ be the number of satellites in space at time t. Give an expression for the probability distribution of $S(6)$?

This part is an application of the infinite-server queue, i.e., the $M_t/GI/\infty$ model, as in the "physics" paper or Chapter 2 of Ross. We know that $S(t)$ has a Poisson distribution for each t, where the mean function is

$$m(t) = \int_0^t \lambda(u) G^c(t-u)\, du = \int_0^t 2u e^{-(1/2)(t-u)}\, du, \quad t \geq 0.$$

Hence,

$$P(S(6) = k) = \frac{e^{-m(6)} m(6)^k}{k!},$$

where

$$m(6) = \int_0^6 2u e^{-(1/2)(6-u)}\, du. \tag{6}$$

(c) Let $R(t)$ be the number of satellites that have been launched and have returned to earth in $[0,t]$. Give an expression for the joint probability $P(S(6) = 7, R(6) = 8)$.

From Theorem 1 of the physics paper (and the Poisson random measure representation there), we know that $S(6)$ and $R(6)$ are independent random variables with means $E[S(6)] = m(6)$ given in (6) above and

$$E[R(t)] = \int_0^t \lambda(u) G(t-u)\, du \quad \text{for} \quad t = 6.$$

However, we know that $S(6) + R(6) = N(6)$, the total number of satellites launched in $[0,6]$. Since $E[N(6)] = 6^2 = 36$, by the reasoning in parts (a) and (b), we can write $E[R(t)] = 36 - E[S(t)]$. Finally, we have

$$P(S(6) = 7, R(6) = 8) = \left(\frac{e^{-E[S(6)]} E[S(6)]^7}{7!}\right)\left(\frac{e^{-E[R(6)]} E[R(6)]^8}{8!}\right),$$

using the formulas for the means already given.

(d) Give an expression for the covariance $Cov(S(6), S(8))$.

This is explained by Theorem 2 and Figure 3 in the physics paper. From the Poisson measure representation, this covariance coincides with the variance of the number of points in a region bounded by two vertical lines and one 45 degree line, which in turn coincides with the mean number in that region. As a consequence,

$$Cov(S(6), S(8)) = \int_0^6 \lambda(u) G^c(8-u)\, du = \int_0^6 2ue^{-(1/2)(8-u)}\, du.$$
