IEOR 6711: Stochastic Models I

First Midterm Exam, Chapters 1-2, October 7, 2012

There are four questions, each with multiple parts.

Justify your answers; show your work.

1. Poisson Process and Transforms (30 points)

Let $\{N(t) : t \geq 0\}$ be a **Poisson process** with rate (intensity) λ.

(a) Give expressions for: (i) the **probability mass function (pmf)** of $N(t)$, $p_{N(t)}(k) \equiv P(N(t) = k)$; (ii) the **probability generating function (pgf)** of $N(t)$, $\hat{p}_N(t)(z)$; the **moment generating function (mgf)**, $\psi_{N(t)}(u)$ and (iv) the **characteristic function (cf)** of $N(t)$, $\phi_{N(t)}(u)$.

(b) Show how the mgf $\psi_{N(t)}(u)$ can be used to derive the mean and variance of $N(t)$.

(c) Use the mgf $\hat{p}_{N(t)}(z)$ to prove or disprove the claim: If $[a, b]$ and $[c, d]$ are two disjoint subintervals of the positive halfline $[0, \infty)$, then the sum $(N(d) - N(c)) + (N(b) - N(a))$ has a Poisson distribution.

(d) What is the probability $P(N(t)\,\text{is even}) \equiv P(N(t) \in \{2k : k \geq 0\})$? Prove that this probability is always greater than $1/2$ and converges to $1/2$ as $t \to \infty$.

Let $\{X_k : k \geq 1\}$ be a sequence of i.i.d. continuous real-valued random variables with **probability density function (pdf)** $f(x)$, mean m and variance σ^2. Let

$$Y(t) \equiv \sum_{k=1}^{N(t)} X_k, \quad t \geq 0.$$

(e) Give an expression for the cf of $Y(t)$.

(f) Derive the mean and variance of $Y(t)$.

(g) Does there exist a finite random variable L with a non-degenerate probability distribution (such that $P(L = c) \neq 1$ for any c) and constants a and b such that

$$\frac{Y(t) - at}{\sqrt{bt}} \Rightarrow L \quad \text{as} \quad t \to \infty, \tag{1}$$

where $\Rightarrow$ denotes **convergence in distribution**? If so, what are a, b and L?

(h) Give a detailed proof to support your answer in part (g).

2. Conditional Remaining Lifetimes (30 points)

Let a **random lifetime** be represented by a nonnegative continuous random variable X with pdf $f(x)$, **cumulative distribution function (cdf)** $F(x) \equiv \int_0^x f(s)\,ds$ and **complementary cdf (ccdf)** $F^c(x) \equiv 1 - F(x) \equiv P(X > x)$ satisfying $F^c(x) > 0$ for all $x \geq 0$. For $t \geq 0$, the associated **conditional remaining lifetimes** are the random variables $X(t)$ with ccdf

$$F^c(x; t) \equiv P(X(t) > x) \equiv P(X > t + x | X > t), \quad t \geq 0, x \geq 0,$$

and associated pdf $f(x;t)$ (with $F(x;t) \equiv \int_0^x f(s;t)\,ds$). Let $r(t) \equiv f(0;t)$.

(a) Give an explicit expression for $r(t)$ in terms of the pdf f of X.

(b) Suppose that $r(t) = 7$, $t \ge 0$. Does that imply that the pdf f of X is well defined and that we know it? If so, what is it?

(c) Suppose that $r(t) = t$, $t \ge 0$. Does that imply that the pdf f of X is well defined and that we know it? If so, what is it?

Now let X_i, $i = 1, 2$, be two independent random lifetimes defined as above, having pdf's $f_i(x)$, cdf's $F_i(x)$, ccdf's $F_i^c(x)$. Let $X_i(t)$ be the associated conditional remaining lifetimes with cdf's $F_i^c(x;t)$ and pdf's $f_i(x;t)$. Let $r_i(t) \equiv f_i(0;t)$

(d) Prove or disprove:

$$P(X_1 < X_2 | \min\{X_1, X_2\} = t) = \frac{r_1(t)}{r_1(t) + r_2(t)}.$$

Now suppose that $r_1(t) \le r_2(t)$ for all $t \ge 0$. Prove or disprove each of the following statements:

(e) $F_1(t) \le F_2(t) \quad \text{for all} \quad t \ge 0$,

(f) $E[X_1^3] \ge E[X_2^3]$,

(g) $P(X_1 \ge X_2) = 1$).

3. Independent Random Variables. (20 points)

Let $\{X_n : n \ge 1\}$ and $\{Y_n : n \ge 1\}$ be independent sequences of independent random variables with X_n distributed the same as Y_n for all $n \ge 1$ and

$$P(X_n = n) = 1 - P(X_n = 0) = \frac{1}{n} \quad \text{for all} \quad n \ge 1.$$

Let

$$Z_n \equiv X_n Y_n \quad \text{and} \quad D_n \equiv X_n - Y_n, \quad n \ge 1.$$

(a) What are the mean and variance of Z_n?

(b) What is the probability that the sum $Z_1 + Z_2 + \cdots + Z_n$ converges to a finite limit as $n \to \infty$?

(c) What is the probability that the sum $D_1 + D_2 + \cdots + D_n$ converges to a finite limit as $n \to \infty$?

(d) Are there deterministic constants a_n and b_n with $a_n \to \infty$, $b_n \to \infty$ and $a_n/b_n \to \infty$ such that

$$\frac{(Z_1 + \cdots + Z_n) - a_n}{b_n} \Rightarrow L \quad \text{as} \quad n \to \infty,$$

where L is a nondegenerate random variable (defined in problem 1 above) and $\Rightarrow$ again denotes convergence in distribution? If so, specify a random variable L and constants a_n and b_n that do the job.

4. Cars on a Highway Segment During Rush Hour (20 points)

Two car-detection devices have been placed on a segment of a one-way highway without exits or entrances in between. Let $N(t)$ be the number of cars that pass the first detection point during $[0, t]$. Suppose we consider a rush hour period in the morning. Thus, we let $\{N(t) : t \geq 0\}$ be a nonhomogeneous Poisson process with rate function $\lambda(t) = 12t$ over an initial time interval $[0, 6]$. However, the first detection device does not work perfectly. Indeed, each car is detected by the initial detection device only with probability $2/3$, independently of the history up to the time it passes the detection device. However, The second detection device works perfectly.

As a simplifying assumption, assume that the cars do not interact, so that the length of time that the cars remain in the highway segment can be regarded as i.i.d. random variables, independent of the times that they pass the detection device. These times are regarded as random variables, because the cars travel at different speeds. Suppose that the length of time each car remains in the highway segment is uniformly distributed over the interval $[1, 3]$.

(a) What is the expected number of cars that pass the first detection point on the highway during the interval $[0, 5]$ and are detected by the detection device?

(b) What is the probability that precisely 50 cars pass the first detection point on the highway during the interval $[0, 5]$ and are detected by the detection device?

(c) Give a convenient approximate expression for the probability that more than 120 cars pass the first detection point on the highway during the interval $[0, 5]$ and are detected by the detection device.

(d) Let $C(t)$ be the number of cars that are detected by the detection device up to time t and remain in the highway segment at time t. Give an expression for the probability distribution of $C(4)$?

(e) What is the covariance between $C(3)$ and $C(6)$?

(f) Let $D(t)$ be the number of cars that are detected by the detection device and have departed by time t. Give an expression for the joint distribution of $C(4)$ and $D(4)$.