IEOR 6711: Stochastic Models I

First Midterm Exam, Chapters 1-2, October 6, 2013

There are five questions, each with multiple parts.

Justify your answers; show your work.

1. Random Hats (15 points)

At a party n people each come wearing a hat. When they leave, a random hat is assigned to each person, with each hat being equally likely.

(a) What is the expected number of people who leave with the same hat they had when they arrived?

(b) What is the variance of the number of people who leave with the same hat they had when they arrived?

(c) Suppose that this hat-matching experiment is repeated in independent experiments with larger and larger groups, with n people in the n^{th} experiment for all $n \geq 2$. Let N be the number of times among all these experiments that two designated people (assumed to be present in all experiments) both get their own hat back. What is $P(N < \infty)$ and why?

2. Variance Formulas (20 points)

(a) Let X and Y be two real-valued random variables. Assume that $E[X^2] < 1$. Prove or disprove:

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]),$$

where $Var(X|Y)$ is defined by

$$Var(X|Y) \equiv E[(X - E[X|Y])^2|Y].$$

(b) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables, each distributed as X, where $E[X^2] < \infty$. Prove or disprove:

$$Var\left(\sum_{i=1}^{N(t)} X_i\right) = \lambda t Var(X).$$

3. The New Six (6) Subway Line. (35 points)

A new subway line has been added to the West Side for the convenience of Columbia students. It has six stations. There are stations at 86^{th} street (station 1), 96^{th} street (station 2), 106^{th} street (station 3), 116^{th} street (station 4), 126^{th} street (station 5) and 136^{th} street (station 6).

We consider only the northbound subway. A northbound subway arrives at station 1 every 10 minutes. The travel time between successive stations is constant, equal to 2 minutes. Suppose that the subway stations and the subway trains have unlimited capacity and that the time to load and unload passengers can be ignored.

For $1 \leq i \leq 5$, customers arrive at station i to use the northbound subway according to a Poisson process with rate λ_i per minute. Suppose that each customer entering station i gets off at station j with probability $P_{i,j}$, independently of all other customers (where $P_{i,j} > 0$ if and only if $j > i$) and

$$\sum_{j=i+1}^{6} P_{i,j} = 1 \quad \text{for all} \quad i, \quad 1 \leq i \leq 5.$$

(a) Give an expression for the expected number of customers to get on the subway (necessarily going north) at each visit to station i.

(b) Give an expression for the probability generating function of the number of customers to get on the subway at each visit to station i.

(c) Give an expression for the expected value of the sum of the waiting times of all customers to get on the subway at each visit to station i.

(d) Suppose that 8 customers get on the subway at station 1 at one specified time. What is the probability that exactly 3 of these customers had to wait more than 4 minutes before getting on the subway?

(e) Give an expression for the probability that the number of customers getting off the northbound subway at a visit to station 4 is exactly j.

(f) Give an expression for the probability that, simultaneously, the number of customers getting off the northbound subway at a visit to station 4 is j and the number getting off at the next stop, at station 5, is k.

(g) Suppose that $\lambda_i = 12 - 2i$ for all i, $1 \leq i \leq 5$, $P_{i,j} = 1/(6-i)$ for all i and j with $j > i$ ($1 \leq i \leq 5$ and $2 \leq j \leq 6$). Determine a convenient accurate approximation for the probability that the number of customers getting off the northbound subway at one specified visit to station 6 is greater than 130? Is that probability more than 1/20? Why is your approximation justified?

4. modes of convergence (15 points)

Consider a sequence of real-valued random variables $\{X_n : n \geq 1\}$.

(a) Define convergence of X_n to a random variable X in (i) mean square, (ii) in probability and (iii) with probability 1.

(b) Prove or disprove: Convergence of X_n to a random variable X in mean square implies convergence of X_n to a random variable X in probability.

(c) Prove or disprove: Convergence of X_n to a random variable X in mean square implies convergence of X_n to a random variable X with probability 1.

5. peaks (15 points)

Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. random variables with a continuous cdf F. We say that a peak occurs at time n if $X_{n-1} < X_n > X_{n+1}$. Let N_n be the number of peaaks among the first n variables. Prove or disprove:

$$\frac{N_n}{n} \to \frac{1}{3} \quad \text{as} \quad n \to \infty \quad \text{with probability 1.}$$