IEOR 6711: Stochastic Models I

Second Midterm Exam, Chapters 3 & 4, November 21, 2010

Justify your answers; show your work.

1. Random Walk on a Graph (25 points)

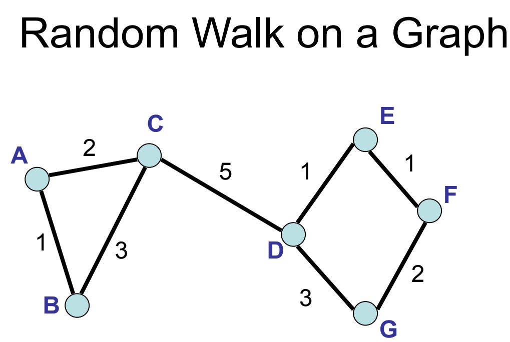

Figure 1: A random walk on a graph.

Consider the graph shown in Figure 1 above. There are 7 nodes, labelled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc. Thus the probability of going from node C to node A in one step is $2/(2+3+5) = 2/10 = 1/5$, while the probability of moving from node C to node B in one step is $3/10$.

(a) What is the long-run proportion of all transitions that are transitions into node A?

(b) Starting from node A, what is the expected number of steps required to return to node A?

(c) Give an expression for the expected number of visits to node G, starting in node A, before going to either node B or node F.

(d) Give an expression for the probability of going to B before going to node F, starting in node A.

Note: A numerical answer is desired in parts (a) and (b), but not in parts (c) and (d).

2. Patterns in Rolls of a Die (20 points)

Consider successive independent rolls of a six-sided die, where each of the sides numbered 1 through 6 is equally likely to appear. ("Die" is the singular of "dice.")

(a) What is the expected number of rolls after the pattern $(1, 1, 2, 1)$ first appears until it appears again?

(b) What is the expected number of rolls until the pattern $(1, 1, 2, 1)$ first appears?

(c) What is the probability that the pattern $(1, 1, 2, 1)$ occurs before the pattern $(1, 1, 1)$?

3. A Computer with Three Parts (30 points)

A computer has three parts, each of which is needed for the computer to work. The computer runs continuously as long as the three required parts are working. The three parts have mutually independent exponential lifetimes before they fail. The expected lifetime of parts 1, 2 and 3 are 10 weeks, 20 weeks and 30 weeks, respectively. When a part fails, the computer is shut down and an order is made for a new part of that type. When the computer is shut down (to order a replacement part), the remaining two working parts are not subject to failure. The time required to receive an order for a new part of type 1 is exponentially distributed with mean 1 week; the time required for a part of type 2 is uniformly distributed between 1 week and 3 weeks; and the time required for a part of type 3 has a gamma distribution with mean 3 weeks and standard deviation 10 weeks.

(a) What is the long-run proportion of time that the computer is working?

(b) What is the long-run proportion of time that the computer is *not* working, and waiting for an order of a new part 1?

(c) State a theorem implying that the probability that the computer is working at time t converges as $t \to \infty$ to a limit equal to the long-run proportion in part (a). Explain why the theorem applies.

(d) State the key renewal theorem and show that it implies the theorem used in part (c).

Do ONE and ONLY ONE of the following two problems: (Problem 5 is judged to be harder, and so potentially worth more.)

4. Wald's equation. (25 points)

(a) State the theorem expressing Wald's equation for i.i.d. random variables.

(b) Prove the theorem in part (a).

(c) State the elementary renewal theorem and show how Wald's equation is applied to prove it.

5. A Ticket Booth. (30 points)

Suppose that customers arrive at a single ticket booth according to a Poisson process with rate λ per minute. Customers are served one at a time by a single server. There is unlimited waiting space. Assume that all potential customers join the queue and wait their turn. (There is no customer abandonment.) Let the successive service times at the ticket booth be mutually independent, and independent of the arrival process, with a cumulative distribution function G having density g and mean $1/\mu$. Let $Q(t)$ be the number of customers at the ticket booth at time t, including the one in service, if any.

(a) Identify random times T_n, $n \geq 1$, such that the stochastic process $\{X_n : n \geq 1\}$ is an irreducible aperiodic discrete-time Markov chain (DTMC), when $X_n = Q(T_n)$ for $n \geq 1$. Determine the transition probabilities for this DTMC.

(b) Find conditions under which $X_n \Rightarrow X$ as $n \to \infty$, where $\Rightarrow$ denotes convergence in distribution and X is a proper random variable, and characterize the distribution of X.

(c) How does the steady-state distribution determined in part (b) simplify when the service-time cdf is $G(x) \equiv 1 - e^{-\mu x}$, $x \geq 0$?

(d) Derive an approximation for the distribution of the steady-state queue content X with general service-time cdf found in part (b), obtained by considering the behavior as the arrival rate λ is allowed to increase.