IEOR 6711: Stochastic Models I

Second Midterm Exam, Chapters 3 & 4, November 21, 2010

SOLUTIONS

Justify your answers; show your work.

1. Random Walk on a Graph (25 points)

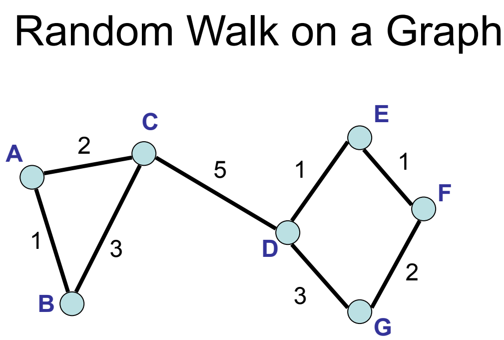

Figure 1: A random walk on a graph.

Consider the graph shown in Figure 1 above. There are 7 nodes, labelled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc. Thus the probability of going from node C to node A in one step is $2/(2+3+5) = 2/10 = 1/5$, while the probability of moving from node C to node B in one step is $3/10$.

(a) What is the long-run proportion of all transitions that are transitions into node A?

The first two parts concerns reversibility, §4.7 of Ross. See pages 205-206. By Theorem 4.7.1, the steady-state probability for each node is proportional to the sum of the weights on the arcs out of that node. Thus, $\pi_A = 3/36 = 1/12$. It is important two divide by twice the sum of the arc weights, because each arc weight is counted twice, once for each of its connecting nodes.

(b) Starting from node A, what is the expected number of steps required to return to node A?

The successive visits to node A also make a renewal process. By the SLLN for renewal processes, the long-run proportion of visits to node A is also the reciprocal of the expected return time. Hence, the expected time to return is the reciprocal of the long-run proportion. Hence the expected time to return to node A is

$$\frac{1}{\pi_A} = \frac{12}{1} = 12.$$

(c) Give an expression for the expected number of visits to node G, starting in node A, before going to either node B or node F.

The last two parts involve an absorbing Markov chain, as in liberating Markov mouse. This topic is also covered in §4.4 of the book. Both approaches lead to solving a system of equations or, equivalently, inverting a matrix. Starting from scratch, you could set up the appropriate system of equations. We display the resulting solution, via the fundamental matrix of the absorbing chain (related to the square submatrix of transition probabilities for the transient states).

As discussed in the lecture notes, here the absorbing DTMC has the general form:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} ,$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 2×2, R is 5×2 and Q is 5×5. The matrix Q describes the probabilities of motion among the transient states, as discussed. The matrix R gives the probabilities of absorption in one step (going from one of the transient states to one of the absorbing states in a single step). Here the absorbing states are nodes B and F. In general Q would be square, say m by m, while R would be m by k, and I would be k by k. In particular,

$$Q = \begin{array}{c} A \\ C \\ D \\ E \\ G \end{array} \begin{pmatrix} 0 & 2/3 & 0 & 0 & 0 \\ 2/10 & 0 & 5/10 & 0 & 0 \\ 0 & 5/9 & 0 & 1/9 & 1/9 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 3/5 & 0 & 0 \end{pmatrix} ,$$

and

$$R = \begin{array}{c} A \\ C \\ D \\ E \\ G \end{array} \begin{pmatrix} 1/3 & 0 \\ 3/10 & 0 \\ 0 & 1/9 \\ 0 & 1/5 \\ 0 & 2/5 \end{pmatrix} ,$$

The column labels for Q are the same as the row labels for Q, whereas the column labels of R

are B and F, in that order.

$$I - Q = \begin{array}{c} A \\ C \\ D \\ E \\ G \end{array} \begin{pmatrix} 1 & -2/3 & 0 & 0 & 0 \\ -2/10 & 1 & -5/10 & 0 & 0 \\ 0 & -5/9 & 1 & -1/9 & -1/9 \\ 0 & 0 & -1/2 & 1 & 0 \\ 0 & 0 & -3/5 & 0 & 1 \end{pmatrix} .$$

Then the matrix N is the inverse of the matrix $I - Q$ above. The answer then is $N_{A,G}$, where $N = (I - Q)^{-1}$, where A and G are the appropriate states. That is the element in the first row and the last column.

(d) Give an expression for the probability of going to B before going to node F, starting in node A.

Here the answer is $B_{A,B}$, where $B = NR$, and A and B are the state labels. It is the row of N corresponding to the initial state A multiplied by the column of R corresponding to the ending absorbing state B.

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2. Patterns in Rolls of a Die (20 points)

This is a minor variant of problem 1 on the 2008 midterm exam. The second pattern in part (c) has been changed. The occurrences of the pattern makes a delayed renewal process; see Example 3.5(A) on p. 125.

Consider successive independent rolls of a six-sided die, where each of the sides numbered 1 through 6 is equally likely to appear. ("Die" is the singular of "dice.")

(a) What is the expected number of rolls after the pattern $(1, 1, 2, 1)$ first appears until it appears again?

For any $n \geq 4$, the probability that the pattern $(1, 1, 2, 1)$ appears at n is $(1/6)^4 = 1/1296$. Thus by the SLLN for a renewal process, the expected return time is the reciprocal of this steady-state probability, i.e., 1296. (The reasoning is the same as in problem 1(b).)

(b) What is the expected number of rolls until the pattern $(1, 1, 2, 1)$ first appears?

$$\begin{aligned} E[N_{1,1,2,1}] &= E[N_1] + E[N_{1 \to 1,1,2,1}] \\ &= E[N_{1 \to 1}] + E[N_{1,1,2,1 \to 1,1,2,1}] \\ &= 6 + 1296 = 1302 \end{aligned}$$

(c) What is the probability that the pattern $(1, 1, 2, 1)$ occurs before the pattern $(1, 1, 1)$?

By reasoning above,

$$\begin{aligned} E[N_{1,1,1}] &= E[N_1] + E[N_{1\to 1,1}] + E[N_{1,1\to 1,1,1}] \\ &= E[N_{1\to 1}] + E[N_{1,1\to 1,1}] + E[N_{1,1,1\to 1,1,1}] \\ &= 6 + 36 + 216 = 258 \end{aligned}$$

Then $E[N_{1,1,1\to 1,1,2,1}] = E[N_{1,1\to 1,1,2,1}]$ and $E[N_{1,1,2,1\to 1,1,1}] = E[N_{1\to 1,1,1}]$, while

$$E[N_{1,1\to 1,1,2,1}] = E[N_{1,1,2,1}] - E[N_{1,1}] = 1302 - 42 = 1260$$

and

$$E[N_{1\to 1,1,1}] = E[N_{1,1,1}] - E[N_1] = (6 + 36 + 216) - 6 = 252.$$

Finally, letting A be the event that pattern $1,1,2,1$ occurs before pattern $1,1,1$, we have from p. 127 of Ross,

$$\begin{aligned} P(A) &= \frac{E[N_{1,1,1}] + E[N_{1,1,2,1\to 1,1,1}] - E[N_{1,1,2,1}]}{E[N_{1,1,1\to 1,1,2,1}] + E[N_{1,1,2,1\to 1,1,1}]} \\ &= \frac{258 + 1260 - 1302}{1260 + 252} = \frac{216}{1512} = \frac{1}{7} \end{aligned}$$

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2. A Computer with Three Parts (30 points)

A computer has three parts, each of which is needed for the computer to work. The computer runs continuously as long as the three required parts are working. The three parts have mutually independent exponential lifetimes before they fail. The expected lifetime of parts 1, 2 and 3 are 10 weeks, 20 weeks and 30 weeks, respectively. When a part fails, the computer is shut down and an order is made for a new part of that type. When the computer is shut down (to order a replacement part), the remaining two working parts are not subject to failure. The time required to receive an order for a new part of type 1 is exponentially distributed with mean 1 week; the time required for a part of type 2 is uniformly distributed between 1 week and 3 weeks; and the time required for a part of type 3 has a gamma distribution with mean 3 weeks and standard deviation 10 weeks.

This is a variant of problem 1 in the lecture notes of October 7. This can be viewed as an alternating renewal process. For the long-run proportions, we can invoke the renewal reward theorem, Theorem 3.6.1 (i). For part (c), we can invoke Theorem 3.4.4.

(a) What is the long-run proportion of time that the computer is working?

The successive times that the computer is working and shut down form an alternating renewal process. Let T be a time until a failure (during which the computer is working) and let D be a down time. Then the long-run proportion of time that the computer is working is $ET/(ET + ED)$.

The random variable T is exponential with rate equal to the sum of the rates; i.e.,

$$ET = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1/10) + (1/20) + (1/30)} = \frac{1}{(11/60)} = \frac{60}{11} \text{ weeks} .$$

Let N be the index of the first part to fail. Since the failure times are mutually independent exponential random variables,

$$P(N = i) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{(i/10)}{(1/10) + (1/20) + (1/30)};$$

e.g.,

$$P(N = 1) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{(1/10)}{(1/10) + (1/20) + (1/30)} = \frac{6}{11} .$$

Now we need to consider the random times it takes to get the replacement parts. The distributions beyond their means do not affect the answers to the questions asked here. Only the means matter here. To find ED, we consider the three possibilities for the part that fails:

$$\begin{aligned} ED &= P(N = 1)E[D|N = 1] + P(N = 2)E[D|N = 2] + P(N = 3)E[D|N = 3] \\ &= (6/11)E[D|N = 1] + (3/11)E[D|N = 2] + (2/11)E[D|N = 3] \\ &= (6/11)1 + (3/11)2 + (2/11)3 \\ &= 18/11 . \end{aligned}$$

Hence,

$$\frac{ET}{ET + ED} = \frac{(60/11)}{(60/11) + (18/11)} = \frac{60}{78} = \frac{10}{13} \approx 0.769$$

(b) What is the long-run proportion of time that the computer is *not* working, and waiting for an order of a new part 1?

Again apply renewal theory. In particular, apply the renewal reward theorem. The long-run proportion of time can be found by the renewal reward theorem: We want ER/EC, where ER is the expected reward per cycle and EC is the expected length of a cycle. Here a cycle is an up time plus a down time; i.e., $EC = ET + ED = 78/11$ from the last part. Here $ER = P(N = 1)E(D|N = 1) = (6/11) \times 1 = 6/11$. So

$$\frac{ER}{EC} = \frac{(6/11)}{(78/11)} = \frac{6}{78} = \frac{1}{13} .$$

(c) State a theorem implying that the probability that the computer is working at time t converges as $t \to \infty$ to a limit equal to the long-run proportion in part (a). Explain why the theorem applies.

The theorem is Theorem 3.4.4. The idea is to state it precisely. We have an alternating renewal process with up intervals and down intervals. These random variables have the finite means already determined. The cycle is nonlattice because the up time has a density. The down time is independent of the up time. The sum of two independent random variables, one of which has a density, itself has a density, and so is nonlattice; Theorem 4, p. 146 of Feller II.

(d) State the key renewal theorem and show that it implies the theorem in part (c).

The key renewal theorem is Theorem 3.4.2. The idea is state it precisely, including the conditions, especially the direct Riemann integrability. In §5 of the lecture notes of October 12, we showed that the key renewal theorem implies Theorem 3.4.4. First, the probability the system is working at time t, $P(t)$ satisfies a renewal equation

$$g(t) = h(t) + \int_0^t g(t-s)\, dF(s),$$

where $P(t) = g(t)$ and $h(t) = 1 - H(t)$ with H being the cdf of an up time. Then the claimed limit follows from the key renewal theorem, as shown in the notes. A proof is also given in Ross, who constructs an equation involving the last renewal before time t instead of the renewal equation. The final step in either case starts from the solution to the renewal equation above, namely,

$$g(t) = h(t) + \int_0^t h(t-s)\, dm(s),$$

where $m(t)$ is the renewal function. We have

$$g(t) \to \frac{\int_0^\infty h(t)\, dt}{E[C]} = \frac{E[T]}{E[T] + E[D]},$$

where C is a "cycle," i.e., an interval between renewals. We need to explain that the function h here is indeed d.R.i.

Do ONE and ONLY ONE of the following two problems. (Problem 5 is judged to be harder, and so potentially worth more.)

4. Wald's equation. (25 points)

(a) State the theorem expressing Wald's equation for i.i.d. random variables.

Theorem 3.3.2 on p. 105. It is important to mention that N must be a stopping time, and it is important to define what that means precisely. It is important to include the assumption of finite expectations.

(b) Prove the theorem in part (a).

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p. 105. It is important to include the correct indicator function. It is important to explain precisely how the stopping time property is used.

(c) State the elementary renewal theorem and show how Wald's equation is applied to prove it. ---

See Corollary 3.3.3 on p. 106. See Theorem 3.3.4 and its proof.

5. A Ticket Booth. (30 points)

Suppose that customers arrive at a single ticket booth according to a Poisson process with rate λ per minute. Customers are served one at a time by a single server. There is unlimited waiting space. Assume that all potential customers join the queue and wait their turn. (There is no customer abandonment.) Let the successive service times at the ticket booth be mutually independent, and independent of the arrival process, with a cumulative distribution function G having density g and mean $1/\mu$. Let $Q(t)$ be the number of customers at the ticket booth at time t, including the one in service, if any.

(a) Identify random times T_n, $n \geq 1$, such that the stochastic process $\{X_n : n \geq 1\}$ is an irreducible aperiodic discrete-time Markov chain (DTMC), when $X_n = Q(T_n)$ for $n \geq 1$. Determine the transition probabilities for this DTMC.

(b) Find conditions under which $X_n \Rightarrow X$ as $n \to \infty$, where $\Rightarrow$ denotes convergence in distribution and X is a proper random variable, and characterize the distribution of X.

(c) How does the steady-state distribution determined in part (b) simplify when the service-time cdf is $G(x) \equiv 1 - e^{-\mu x}$, $x \geq 0$?

(d) Derive an approximation for the distribution of the steady-state queue content X with general service-time cdf found in part (b) obtained by considering the behavior as the arrival rate λ is allowed to increase.

This is a minor variant of problem 3 on the 2007 second midterm exam. This is the $M/G/1$ queue, discussed in class on Thursday, November 11; see the class lecture notes.